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Least-Squares Solutions of the Matrix Equation $A^T XA = B$ Over Bisymmetric Matrices and its Optimal Approximation

Yanyan Zhang*
(*Hunan University of Arts and Science, Changde 415000, China
E-mail: dingdang8142@163.com*)

Yuan Lei and Anping Liao
(*College of Mathematics and Econometrics, Hunan University, Changsha 410082, China
E-mail: e_ylei@yahoo.com.cn, lap@ccsu.cn*)

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Abstract

A real $n \times n$ symmetric matrix $X = (x_{ij})_{n\times n}$ is called a bisymmetric matrix if $x_{ij} = x_{n+1-j,n+1-i}$. Based on the projection theorem, the canonical correlation decomposition and the generalized singular value decomposition, a method useful for finding the least-squares solutions of the matrix equation $A^T XA = B$ over bisymmetric matrices is proposed. The expression of the least-squares solutions is given. Moreover, in the corresponding solution set, the optimal approximate solution to a given matrix is also derived. A numerical algorithm for finding the optimal approximate solution is also described.

Keywords: Bisymmetric matrix; canonical correlation decomposition; generalized singular value decomposition; least-squares solution; optimal approximate solution.
Mathematics subject classification: 15A24, 65F20, 65F30

1. Introduction

Let $R^{n\times m}$ denote the set of all real $n\times m$ matrices, $SR^{n\times n}$ ($ASR^{n\times n}$) the set of all real symmetric (anti-symmetric) matrices, $BSR^{n\times n}$ the set of all real bisymmetric matrices, $OR^{n\times n}$ the set of all real orthogonal matrices, I_n the identity matrix in $R^{n\times n}$, S_n the antitone identity matrix in $R^{n\times n}$, namely, $S_n = (e_n, e_{n-1}, \ldots, e_1)$, where e_i denotes the ith column of I_n. $\|.\|$ stands for the Frobenius norm. $A * B$ represents the Hadamard product of two $n \times m$ matrices A and B, that is, $A * B = (a_{ij} b_{ij})$, $1 \le i \le n, 1 \le j \le m$.

Matrix equation is one of the important study fields of linear algebra. The matrix equation

$$A^T XA = B \tag{1.1}$$

comes from an inverse problem of vibration theory. Dai and Lancaster [1] have studied the least-squares problem of (1.1). The expression of the least-squares solution of the matrix equation (1.1) over symmetric matrices, semidefinite symmetric matrices and bisymmetric matrices were given in [1] and [5], which used the singular value decomposition (SVD)

*Corresponding author.

and the canonical correlation decomposition (CCD). Xie [3] considered the least-squares solutions of the matrix equation (1.1) over semidefinite (but not be symmetric) matrices and its optimal approximate solutions. Based on the results in [5], which gave the expression of least-squares solutions of the matrix equation (1.1) over bisymmetric matrices, we will study the problem of its optimal approximation. That is to say we have two problems to solve in this paper:

- Problem I: For given matrices $A \in R^{n\times m}$, $B \in SR^{m\times m}$, find a matrix $\widetilde{X} \in BSR^{n\times n}$, such that
$$\|A^T\widetilde{X}A - B\| = \min_{X\in BSR^{n\times n}} \|A^TXA - B\|.$$
- Problem II: For given $X^* \in SR^{n\times n}$, find a matrix $\widehat{X} \in S_E$ such that
$$\|\widehat{X} - X^*\| = \min_{X\in S_E} \|X - X^*\|,$$
where S_E denotes the set of solutions of Problem I.

Liao [5] gave the expression of the solutions of Problem I by applying the canonical correlation decomposition (CCD). But because a general nonsingular matrix in CCD does not satisfy the orthogonal invariance of the Frobenius norm, we cannot obtain the solution of Problem II through the solutions of Problem I. In order to solve this problem, we can use the given solution X_0 and the projection theorem to transform Problem I to another problem which is to find bisymmetric solutions of a consistent equation. Then by applying the generalized singular value decomposition (GSVD) we can obtain the bisymmetric solutions of the consistent equation and its optimal approximation.

In Problem I, if the matrix B is not symmetric, then we can use
$$\begin{aligned}\|A^TXA - B\|^2 &= \|A^TXA - (S(B) + R(B))\|^2\\ &= \|A^TXA - S(B)\|^2 + \|R(B)\|^2,\end{aligned}$$
where $S(B) = \frac{1}{2}(B + B^T) \in SR^{m\times m}$, $R(B) = \frac{1}{2}(B - B^T) \in ASR^{m\times m}$. Therefore, the matrix equation (1.1) and the matrix equation $A^TXA = S(B)$ have the same least-squares bisymmetric solutions. Therefore, we can assume the matrix B in Problem I belongs to $SR^{m\times m}$, and assume the matrix X^* in Problem II satisfies $X^* \in SR^{n\times n}$ for the same reason.

2. Several lemmas and CCD, GSVD

Lemma 2.1. ([6]) *When $n = 2k$,*
$$BSR^{n\times n} = \left\{ \begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix} \middle| M, H \in SR^{n\times n} \right\},$$
and when $n = 2k+1$,
$$BSR^{n\times n} = \left\{ \begin{pmatrix} N & c & HS_k \\ c^T & \rho & c^TS_k \\ S_kH & S_kc & S_kNS_k \end{pmatrix} \middle| N, H \in SR^{k\times k}, c \in R^k, \rho \in R^1 \right\}.$$