NUMERICAL MATHEMATICS, A Journal of Chinese Universities (English Series)
Numer. Math. J. Chinese Univ. (English Ser.), issue 2, vol. **16**: 97-111, 2007

Finite Difference Method for Reaction-Diffusion Equation with Nonlocal Boundary Conditions

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Received May 14, 2005; Accepted (in revised version) November 5, 2006

Abstract

In this paper, we present a numerical approach to a class of nonlinear reaction-diffusion equations with nonlocal Robin type boundary conditions by finite difference methods. A second-order accurate difference scheme is derived by the method of reduction of order. Moreover, we prove that the scheme is uniquely solvable and convergent with the convergence rate of order two in a discrete L_2-norm. A simple numerical example is given to illustrate the efficiency of the proposed method.

Keywords: Reaction-diffusion; nonlocal Robin type boundary; finite difference; solvability; convergence.
Mathematics subject classification: 65M06, 65M12, 65M15

1. Introduction

Reaction-diffusion equations with nonlocal boundary conditions have been given considerable attention in recent years, and various methods have been developed for the treatment of these equations (see [1-9]). Most of the discussions in the current literatures are developed to the Dirichlet type nonlocal boundary conditions problem (see [10-17]), and much less is given to the problem with nonlocal Robin type boundary conditions (see [18]). The purpose of this article is to give a numerical treatment to a class of reaction-diffusion equations with nonlocal boundary conditions by finite difference method. The system of equations to be considered is as follows

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(a(x,t)\frac{\partial u}{\partial x}\Big) + b(x,t)\frac{\partial u}{\partial x} + f(u,x,t), \qquad 0 < x < 1, 0 < t \le T, \tag{1.1a}$$

$$a(0,t)\frac{\partial u}{\partial x}(0,t) - \sigma_1(t)u(0,t) = \int_0^1 \alpha(s)u(s,t)ds + g_1(t), \qquad 0 \le t \le T, \tag{1.1b}$$

$$a(1,t)\frac{\partial u}{\partial x}(1,t) + \sigma_2(t)u(1,t) = \int_0^1 \beta(s)u(s,t)ds + g_2(t), \qquad 0 \le t \le T, \tag{1.1c}$$

$$u(x,0) = \varphi(x), \qquad 0 \le x \le 1, \tag{1.1d}$$

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where u is an unknown function, $a,\ b,\ f,\ \sigma_j,\ g_j (j=1,2),\ \alpha,\ \beta$ and φ are given functions.

By the influence of Robin type boundary, it is not easy to derive a higher accurate scheme. In [18], the author supposed a and b are independent of t and developed a nonlinear monotone iterative difference scheme of (1.1) using the method of upper and lower solutions. However, the truncation error is only $\mathcal{O}(\tau+h)$, and the proof of convergence is not provided. It is also noticed that the proofs of numerical methods in all the recent articles including [18] use the conditions

$$\|\alpha\|_{L_1([0,1])} \equiv \int_0^1 |\alpha(s)|ds < 1, \quad \|\beta\|_{L_1([0,1])} \equiv \int_0^1 |\beta(s)|ds < 1. \tag{1.2}$$

In this article, we develop a linear difference scheme by the method of reduction of order (see [19,20]), and prove the difference scheme is uniquely solvable and of second order rate of convergence in L_2-norm by the energy method. In our proof, it is found that the condition (1.2) is not necessary.

Throughout this article, we suppose that problem (1.1) has a unique smooth solution $u(x,t) \in C^{(4,3)}_{x,t}(\Omega_T)$, where $(x,t) \in \Omega_T \equiv \{0 \le x \le 1,\ 0 \le t \le T\}$. In addition, the following basic conditions are always assumed: when $|\ \varepsilon_i\ | \le \varepsilon_0,\ i=1,2$ and $(x,t) \in Q_T$, we have

$$\Big| f(u(x,t)+\varepsilon_1,x,t) - f(u(x,t)+\varepsilon_2,x,t) \Big| \le c_1 \mid \varepsilon_1 - \varepsilon_2 \mid, \tag{1.3}$$

$$c_0 \le a(x,t) \le c_1,\ \ |\ b(x,t)\ | \le c_1,\ \ |\ \sigma_1(t)\ | \le c_1,\ \ |\ \sigma_2(t)\ | \le c_1. \tag{1.4}$$

where $c_0,\ c_1$ and ε_0 are positive constants.

The outline of the article is as follows. In Section 2, a difference scheme for (1.1) is derived and the finite difference system is tri-diagonal at each time level, which can be solved by Thomas' algorithm. In Section 3, it is proved that the difference scheme is uniquely solvable and of second order rate of convergence in L_2-norm. Finally, a numerical example is given to verify the validity of the analytic results.

2. Difference scheme

In order to approximate the boundary condition, we give some lemmas at first.

Lemma 2.1. ([21]) *Let M be an integer, $h=1/M$, and $x_{i+\frac{1}{2}}=(i+\frac{1}{2})h,\ 0 \le i \le M-1$. If $g(x) \in C^2[0,1]$, then*

$$\int_0^1 g(x)dx - h\sum_{i=0}^{M-1} g(x_{i+\frac{1}{2}}) = \frac{1}{24}h^2 \frac{d^2 g(x)}{dx^2}\bigg|_{x=\eta}, \quad \eta \in (0,1).$$

Lemma 2.2. *Let M be an integer, $h=1/M,\ x_i=ih,\ 0 \le i \le M;\ x_{i+\frac{1}{2}}=(i+\frac{1}{2})h,\ 0 \le i \le M-1$. If $f(x),\ g(x) \in C^2[0,1]$, then*

$$\int_0^1 f(x)g(x)dx - \frac{h}{2}\sum_{i=0}^{M-1} f(x_{i+\frac{1}{2}})\Big[g(x_i)+g(x_{i+1})\Big] = \mathcal{O}(h^2).$$