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# On Non-Uniform Algebraic-Hyperbolic (NUAH) B-Splines

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**Abstract.** The recurrence algorithm is given for the calculation of NUAH B-splines in the space $S_{n+1} = \text{span}\{\sinh t,\ \cosh t,\ t^{n-3}, \cdots,\ t^2, t, 1\}$ $(n \geq 3)$. The case of NUAH B-spline bases of low order with multiple knot sequences is studied. The limiting cases of UAH B-splines are recovered when shape parameters $\alpha' s \to 0^+$and $+\infty$. Then the corresponding NUAH B-spline curve is defined and its main properties such as shape-preserving properties are investigated.

**Key words**: Non-uniform algebraic-hyperbolic B-spline; uniform B-spline; variation diminishing; convexity preserving; shape-preserving.

**AMS subject classifications:** 41A15, 65D07

## 1 Introduction

The NURBS scheme has become a de facto standard in CAGD, which is based on algebraic polynomials, see [1-3]. It is a powerful tool for constructing free-form curves and surfaces. However, rational curves need additional parameters, namely the weights for each control point. So the scheme has several drawbacks such as curves of very high degree after repeated differentiation. Therefore, we would like to find new alternative to the rational model for constructing fair-shape-preserving approximations that inherit major geometric properties such as positivity, monotonicity, convexity, etc. It is also noticed that several methods have been proposed in the space of mixed algebraic and non-algebraic polynomials, see [4-11].

Pottman and Wagner [4] investigated helix splines. GB-splines were constructed for tension generalized splines allowing the tension parameters to vary interval to interval in [5] and the main results were expanded to GB-splines of arbitrary order in [6]. Zhang [7, 8] constructed C-B splines of order based on the space $\text{span}\{\sin t, \cos t,\ t, 1\}$. Wang [9] studied non-uniform algebraic-trigonometric B-splines $k$ $(k \geq 3)$ for the space $\text{span}\{1, t, t^2, \cdots,\ t^{k-3}, \cos t,\ \sin t\}$. Lü constructed uniform hyperbolic polynomial B-splines for the space $\{1, t, t^2, \cdots,\ t^{k-3}, \cosh t, \sinh t\}$ $(k \geq 3)$ in [10].

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In this work, we propose a new kind of splines, namely non-uniform algebraic-hyperbolic (NUAH) B-splines. The local support basis functions are constructed for the space

$$S_{n+1} = \text{span}\{\sinh t, \cosh t, t^{n-3}, \cdots, t^2, t, 1\} \quad (n \geq 3).$$

Studies show that the NUAH B-splines are the expansion of the uniform hyperbolic polynomial (UAH: uniform algebraic-hyperbolic) B-splines. The limits of UAH B-splines show the relation between them and the uniform polynomial B-splines.

The remaining part of the paper is organized as follows. We construct the NUAH B-spline functions and give the recurrence algorithm for the calculation of the splines in Section 2. We then study NUAH B-spline bases of low order with multiple knot sequences in Section 3. In Section 4, we deduce the recurrence formulas of the UAH B-splines and recover the limiting cases when shape parameters $\alpha' s \to 0^+, +\infty$. Then we define the corresponding NUAH B-spline curve and investigate the shape-preserving properties such as variation diminishing (V. D.) property and convexity preserving property in Section 5.

# 2 The basis of NUAH B-splines

## 2.1 NUAH spline space

Let $x = \{x_i\}_{i=-1}^{n+1}(x_{i+1} - x_i > 0)$ partition the parameter axis $x$ in $\mathbf{R}$ (or $[a, b]$), where $x_{-1} = -\infty$ (or, $a$), $x_{n+1} = +\infty$ (or, $b$). $M = \{m_i\}_{i=0}^{n}(1 \leq m_i \leq k)$ is the multiplicity sequence of $x_i(i = 0, 1, \cdots, n)$. $T = \{t_0, t_1, \cdots, t_m\}$ is the multiple knot sequence in $S_k$, where $m = \sum_{j=0}^{n} m_j, S_k =$span$\{\sinh t, \cosh t, t^{k-3}, \cdots, t, 1\}$ $(k \in \mathbf{N}, k \geq 3)$ and $S_2 =$span$\{\sinh t, \cosh t\}$.

**Remark 2.1.** If $f \in S_{2k}(k \geq 2)$, then $(L^*L)(f) = 0$, where $L = D^{k-1}(D - 1)$, that is, $S_k$ is the zero space of $L^*L$.

**Definition 2.1.** Denote $\Omega_{k,m} = \Omega_{k,m}(S_k; M; x) = \{f : f|_{(x_i, x_{i+1})} \in S_k, i = -1, 0, \cdots, n, D^{j-1}f(x_i^-) = D^{j-1}f(x_i^+), j = 1, \cdots, k - m_i, i = 0, \cdots, n\}$. Then $\Omega_{k,m}$ is called the space of non-uniform algebraic-hyperbolic (NUAH) spline of order $k$ with the multiplicity sequence $M$ and the partition $x$.

**Remark 2.2.** As $M = \{1, 1, \cdots, 1\}, \Omega_{k,m} = \Omega_{k,n+1}(S_k; M; x) = \Omega_{k,n+1}(S_k; x) = \{f : f|_{(x_i, x_{i+1})} \in S_k, i = -1, 0, \cdots, n, f \in C^{k-2}(\mathbf{R})$(or, $[a, b]$)$\}$.

**Remark 2.3.** As $M = \{k, k, \cdots, k\}, \Omega_{k,n+1}(S_k; M; x)$ has breakpoints at $t = x_i$.

From Definition 2.1, we need $k(n+2)$ conditions to get a function $f \in \Omega_{k,m}$. However, taking into account the continuity condition in $\Omega_{k,m}$, we can get $k(n + 1) - m$ conditions. Hence, we have

**Lemma 2.1.**

$$\dim\Omega_{k,m}(S_k; M; x) = m + k.$$

**Lemma 2.2.** *For any $f \in \Omega_{k,n+1}(S_k; M; x)$ $\{0\}$, we have*

$$Z(f) \leq k - 1 + m,$$

*where $Z(f)$ is the number of zero points in $\mathbf{R}$ (or on the interval $[a, b]$).*