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Explicit One-Step P-Stable Methods for Second Order Periodic Initial Value Problems†

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Abstract. In this paper, we present an explicit one-step method for solving periodic initial value problems of second order ordinary differential equations. The method is P-stable, and of first algebraic order and high phase-lag order. To improve the algebraic order, we give a composition second order scheme with the proposed method and its adjoint. We report some numerical results to illustrate the efficiency of our methods.

Key words: Periodic initial value problem; the interval of periodicity; P-stable; phase-lag order.

AMS subject classifications: 65L05

1 Introduction

Let us consider the initial value problem of second order ordinary differential equations having periodic solutions

$$y'' = f(x, y), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0 \tag{1}$$

where presume $f(x, y) : [x_0, T] \times R^m \to R^m$ is sufficiently differentiable and in which the first derivative does not appear explicitly. The numerical methods for solving (1) have been paid much attention to in recent years because such problems often arise in elastodynamics, structural mechanics, celestial mechanics, quantum mechanics, seismology or when the hyperbolic PDEs are semidiscretized in space.

In 1976, Lambert and Watson [9] proposed the conceptions of periodicity interval and P-stability which are used to discuss the stability behavior of the numerical methods for second order periodic initial value problems. Moreover, the phase-lag error is also an important character of these numerical methods. So, those numerical methods which are P-stable or of long interval of periodicity, and of high phase-lag error order are desirable when we solve problem (1). For details, we refer to [1,4,9,13,15] and the references therein. Many conventional P-stable methods

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have been proposed, such as linear multi-step methods, high order hybrid P-stable methods and so on [2,3,5-7,10,11,13,14], however, these methods are almost implicit, so for nonlinear problems, a nonlinear algebraic subsystem has to be solved at each step. Indeed, a result due to Lambert and Watson is that explicit linear multi-step methods are not P-stable and the order of P-stable linear multi-step methods is only two at most [9].

Note the stability and phase-lag error analysis are based on the scalar model equation

$$y'' = -\omega^2 y, \qquad \omega \in \mathbf{R}^+, \tag{2}$$

so the following quotient

$$Q = \frac{h^2 f_n}{y_n}$$

may be useful in constructing some one-step methods for solving (1), because for the test equation (2), $Q = -(h\omega)^2 = -\nu^2$, where h is the integration step, $\nu = h\omega$, y_n is the approximation to $y(x_n)$, $f_n = f(x_n, y_n)$, and we suppose $y_n \neq 0$. The aim of this paper is to present a nonconventional explicit one-step method for solving (1) by exploiting the above quotient. The method is P-stable, of first algebraic order and high phase-lag order. To improve the algebraic order, we give a composition second order scheme with the proposed method and its adjoint method.

For completeness, we give some preliminaries for the stability and phase-lag error analysis of numerical methods for solving (1). When some one-step method is applied to the test equation (2), one can obtain the difference equation of the form

$$\begin{pmatrix} y_{n+1} \\ hy'_{n+1} \end{pmatrix} = R(\nu^2) \begin{pmatrix} y_n \\ hy'_n \end{pmatrix}, \tag{3}$$

where $R(\nu^2)$ is a second order square matrix named *stability matrix* only dependent on ν^2.

Definition 1.1. Let λ_1 and λ_2 be the eigenvalues of $R(\nu^2)$. The method is of *the interval of periodicity* $(0, \Gamma^2)$, if λ_1 and λ_2 are conjugate complex and $|\lambda_1| = |\lambda_2| = 1$, for all $\nu^2 \in (0, \Gamma^2)$. The method is *P-stable*, if the interval of periodicity is $(0, +\infty)$.

Let $\det(R)$ and $\mathrm{tr}(R)$ denote the determinant and trace of matrix $R(\nu^2)$. Then it is easily shown that the following result using the Schur condition.

Theorem 1.1. *The method is P-stable if and only if for all $\nu^2 > 0$*

$$\det(R) = 1, \qquad |\mathrm{tr}(R)| \leq 2. \tag{4}$$

Definition 1.2. For a one-step method, the *phase-lag error (dispersion error)* and the *dissipation error(amplification error)*, are defined respectively by

$$\varphi(\nu) = \nu - \cos^{-1}\left(\frac{\mathrm{tr}(R)}{2\sqrt{\det(R)}}\right), \qquad \psi(\nu) = 1 - \sqrt{\det(R)}. \tag{5}$$

Moreover, a method is said to be of *order p phase-lag error* and *order q amplification error*, if

$$\varphi(\nu) = \mathcal{O}(h^{p+1}), \qquad \psi(\nu) = \mathcal{O}(h^{q+1}), \qquad \text{as} \quad h \to 0. \tag{6}$$

If $p = +\infty$, then we call the method *phase-fitted*. If $q = +\infty$, we call the method *zero-dissipative*.

Evidently, a method with finite amplification order is not P-stable, and a zero-dissipative method has $2r$ phase-lag order if and only if $\mathrm{tr}(R(\nu^2)) - 2\cos(\nu) = \mathcal{O}(\nu^{2r+2})$.