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A Class of Constrained Inverse Eigenproblem and Associated Approximation Problem for Symmetric Reflexive Matrices†

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Abstract. Let $S \in R^{n\times n}$ be a symmetric and nontrival involution matrix. We say that $A \in R^{n\times n}$ is a symmetric reflexive matrix if $A^T = A$ and $SAS = A$. Let $SR_r^{n\times n}(S)$=$\{A|A = A^T, A = SAS, A \in R^{n\times n}\}$. This paper discusses the following two problems. The first one is as follows. Given $Z \in R^{n\times m}$ $(m < n)$, $\Lambda = diag(\lambda_1, \cdots, \lambda_m) \in R^{m\times m}$, and $\alpha, \beta \in R$ with $\alpha < \beta$. Find a subset $\varphi(Z, \Lambda, \alpha, \beta)$ of $SR_r^{n\times n}(S)$ such that $AZ = Z\Lambda$ holds for any $A \in \varphi(Z, \Lambda, \alpha, \beta)$ and the remaining eigenvalues $\lambda_{m+1}, \cdots, \lambda_n$ of A are located in the interval $[\alpha, \beta]$. Moreover, for a given $B \in R^{n\times n}$, the second problem is to find $A_B \in \varphi(Z, \Lambda, \alpha, \beta)$ such that

$$\|B - A_B\| = \min_{A\in\varphi(Z,\Lambda,\alpha,\beta)} \|B - A\|,$$

where $\|.\|$ is the Frobenius norm. Using the properties of symmetric reflexive matrices, the two problems are essentially decomposed into the same kind of subproblems for two real symmetric matrices with smaller dimensions, and then the expressions of the general solution for the two problems are derived.

Key words: Symmetric reflexive matrix; constrained inverse eigenproblem; approximation problem; Frobenius norm.

AMS subject classifications: 15A29

1 Introduction

Let $R^{n\times m}$, $SR^{n\times n}$ and $OR^{n\times n}$ denote the sets of real $n \times m$ matrices, real $n \times n$ symmetric matrices and real $n \times n$ orthogonal matrices, respectively, I_n be the identity matrix of dimension n, A^T and $N(A)$ denote the transpose and the null space of a matrix A, respectively. Define matrix inner product $(A, B) = tr(B^T A)$, $A, B \in R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of matrix produced by the inner product is Frobenius norm, i.e., $\|A\| = \sqrt{(A, A)} = (tr(A^T A))^{\frac{1}{2}}$.

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In this paper, $S \in R^{n\times n}$ is a symmetric and nontrival involution matrix; i.e., $S^T = S = S^{-1} \neq \pm I_n$. According to Definition 2.1 in [1], we say that $A \in R^{n\times n}$ is a symmetric reflexive matrix if $A^T = A$ and $SAS = A$, and a vector $x \in R^n$ is said to be reflexive (or antireflexive) if $x = Sx$ (or $x = -Sx$). Let $SR_n^{n\times n}(S)$=$\{A|A = A^T, A = SAS, A \in R^{n\times n}\}$.

If λ is an eigenvalue of $B \in R^{n\times n}$, let $\varepsilon_B(\lambda)$ denote the λ-eigenspace of B. We will say an eigenvalue λ of B to be even (or odd) if $\varepsilon_B(\lambda)$ contains a nonzero reflexive (or antireflexive) vector.

This paper discusses the following two problems:

Problem I Given $Z \in R^{n\times m}$ $(m < n)$, $\Lambda = diag(\lambda_1, \cdots, \lambda_m) \in R^{m\times m}$, and $\alpha, \beta \in R$ with $\alpha < \beta$. Find a subset $\varphi(Z, \Lambda, \alpha, \beta)$ of $SR_r^{n\times n}(S)$ such that $AZ = Z\Lambda$ holds for any $A \in \varphi(Z, \Lambda, \alpha, \beta)$ and the remaining eigenvalues $\lambda_{m+1}, \cdots, \lambda_n$ of A are located in the interval $[\alpha, \beta]$.

Problem II Given $B \in R^{n\times n}$. Find $A_B \in \varphi(Z, \Lambda, \alpha, \beta)$ such that

$$\|B - A_B\| = \min_{A\in\varphi(Z,\Lambda,\alpha,\beta)} \|B - A\|.$$

where $\|.\|$ is the Frobenius norm.

The above two problems arose from structural design. If $S = I_n$, then $SR_r^{n\times n}(S) = SR^{n\times n}$ and Problems I and II are the two questions considered in reference [2]. In this paper, we extend the results in [2] to the class of symmetric reflexive matrices. The main contribution of this paper are using the properties of symmetric reflexive matrices to decompose the Problems I and II into the same kind of subproblems for two real symmetric matrices with smaller dimensions and solving the above two problems.

A symmetric reflexive matrix is also called symmetric ortho-symmetric matrix in [4]. It is necessary to point out that the questions considered in this paper are different from those in reference [4]. In fact, the inverse eigenproblem considered in [4] is a special case of Problem I $(\alpha = -\infty, \beta = +\infty)$. On the other hand, the method adopted in this paper is different from that in reference [4].

In Section 2, we will use the properties of symmetric reflexive matrices to show that Problems I and II can be decomposed into the same kind of subproblems for two real symmetric matrices with smaller dimensions. In Section 3, we will give the general solutions of Problems I and II.

2 Reduction of problems I and II

First we introduce two matrices defined in reference [3]. Let $\{p_1, \cdots, p_s\}$ and $\{q_1, \cdots, q_{n-s}\}$ be orthonormal bases for $\varepsilon_S(1)$ and $\varepsilon_S(-1)$ respectively, and define

$$P = [p_1 \cdots p_s] \in R^{n\times s} \quad and \quad Q = [q_1 \cdots q_{n-s}] \in R^{n\times(n-s)}.$$

Since $S^T = S = S^{-1} \neq \pm I_n$, $1 \leq s < n$ and $[P \;\; Q] \in OR^{n\times n}$. Although P and Q are not unique, finding suitable P and Q is straightforward.

The following lemma characterizes the special structure of the members of $SR_r^{n\times n}(S)$, and it follows immediately from Theorem 1 of reference [3].

Lemma 2.1. *$A \in SR_r^{n\times n}(S)$ if and only if*

$$A = [P \;\; Q] \begin{bmatrix} A_{PP} & 0 \\ 0 & A_{QQ} \end{bmatrix} \begin{bmatrix} P^T \\ Q^T \end{bmatrix}, \quad A_{PP} \in SR^{s\times s}, \quad A_{QQ} \in SR^{(n-s)\times(n-s)}, \qquad (1)$$

where $A_{PP} = P^TAP$ and $A_{QQ} = Q^TAQ$.