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# Algorithms for Finding the Inverses of Factor Block Circulant Matrices†

Zhaolin Jiang[1,*], Zongben Xu[2] and Shuping Gao[3]

[1] *School of Science, Xi'an Jiaotong University, Xi'an 710049, China/Department of Mathematics, Linyi Teachers College, Linyi 276005, China/College of Mathematics, Qufu Normal University, Qufu 273165, China.*
[2] *School of Science, Xi'an Jiaotong University, Xi'an 710049, China.*
[3] *Department of Applied Mathematics, Xidian University, Xi'an 710071, China.*

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**Abstract.** In this paper, algorithms for finding the inverse of a factor block circulant matrix, a factor block retrocirculant matrix and partitioned matrix with factor block circulant blocks over the complex field are presented respectively. In addition, two algorithms for the inverse of a factor block circulant matrix over the quaternion division algebra are proposed.

**Key words**: Inverses matrix; factor block circulant matrix; partitioned matrix; factor block retrocirculant matrix.

**AMS subject classifications:** 15A21, 65F15

## 1 Introduction

Factor block circulant matrices arise in diverse fields of applications [1–3], especially on the differential equations involving circulant matrices. So, computing the inverse of the factor block circulant matrix has become an important problem. In order to solve differential equations involving circulants, we consider in this work the inverses of factor block circulants over the complex field and the quaternion division algebra.

In Section 1, a computation formula for the inverse of a factor block circulant matrix over the complex field is presented by utilizing only the interpolation methods and basic properties of matrix. A remarkable character of the method needs neither the diagonalization method of a factor block circulant matrix nor the theory of the Jordan canonical form.

In Section 2, a computation formula for the inverse of partitioned matrix with factor block circulant blocks over the complex field is presented by using Schur complements.

In Section 3, we consider a new kind of matrices which are factor block circulant matrices over the quaternion division algebra and give a sufficient and necessary condition to determine

*Correspondence to: Zhaolin Jiang, School of Science, Xi'an Jiaotong University, Xi'an 710049, China/Department of Mathematics, Linyi Teachers College, Linyi 276005, China/College of Mathematics, Qufu Normal University, Qufu 273165, China. Email: jzh1208@sina.com
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whether a factor block circulant matrix is singular or not and propose two algorithms for the inverse of a factor block circulant matrix over the quaternion division algebra.

In Section 4, by utilizing only the relationship between a factor block retrocirculant matrix and a factor block circulant matrix, a computation formula for the inverse of a factor block retrocirculant matrix over the complex field is presented.

**Definition 1.1.** Let $C_1, C_2, \cdots, C_m, A$ be square matrices each of order $n$ over the complex field $\mathbb{C}$. We assume that $A$ is nonsingular and that it commutes with each of the $C_k'$s. By an A-factor block circulant matrix of type $(m, n)$ over the complex field $\mathbb{C}$ is meant an $mn \times mn$ matrix of the form

$$\Re = \mathrm{circ}_A(C_1, C_2, \cdots, C_m) = \begin{pmatrix} C_1 & C_2 & \cdots & C_{m-1} & C_m \\ AC_m & C_1 & \cdots & C_{m-2} & C_{m-1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ AC_3 & AC_4 & \cdots & C_1 & C_2 \\ AC_2 & AC_3 & \cdots & AC_m & C_1 \end{pmatrix}.$$

We define $\pi_A$ as the basic $A$- factor circulant over $\mathbb{C}$, that is,

$$\pi_A = \begin{pmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & I \\ A & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

The following useful results are well known [1].

**Lemma 1.1.** *$\Re$ is an $A$- factor block circulant matrix over $\mathbb{C}$ if and only if $\Re = \mathcal{F}(\pi_A)$ for some matrix polynomial $\mathcal{F}(z)$. The polynomial $\mathcal{F}(z) = \sum_{k=0}^{m-1} C_{k+1} z^k$ will be called the representer of the factor circulant over $\mathbb{C}$.*

**Lemma 1.2.** *Two A-factor circulants over $\mathbb{C}$ $\mathcal{B} = circ_A(B_1, \cdots, B_m)$, $\Re = circ_A(C_1, \cdots, C_m)$ commute if the $B_j$'s commute with the $C_j$'s.*

**Lemma 1.3.** *Let $\Re$ be an $A$- factor block circulant over $\mathbb{C}$. Then*

$$\Re = V_A \mathcal{F}(D_A) V_A^{-1},$$

*where*

$$V_A = V_n(K, \omega K, \ldots, \omega^{m-1} K), \quad \mathcal{F}(D_A) = \mathrm{diag}[\mathcal{F}(K), \mathcal{F}(\omega K), \ldots, \mathcal{F}(\omega^{m-1} K)],$$
$$\omega = \exp(2\pi i/m), \quad \mathcal{F}(z) = \sum_{k=0}^{m-1} C_{k+1} z^k.$$

**Lemma 1.4.** *The inverse matrix $\Re^{-1}$ of a nonsingular factor block circulant matrix $\Re$ over $\mathbb{C}$ is also a factor block circulant matrix of the same type.*

**Lemma 1.5.** *Let $K$ denote the principal $m$th root of the nonsingular matrix $A$ over $\mathbb{C}$. Then $V_n(K, \omega K, \ldots, \omega^{m-1} K)$ is nonsingular, and its inverse equals*

$$F_{mn} X^{-1}/\sqrt{m} = \frac{1}{m} [V_n(K^{-1}, \varpi K^{-1}, \ldots, \varpi^{m-1} K^{-1})]^T,$$

*where*

$$X = \mathrm{diag}[I, K, K^2, \ldots, K^{m-1}].$$