English Series | NUMERICAL MATHEMATICS | Vol.14, No.2
A Journal of Chinese Universities | May 2005

ON OPPENHEIM'S INEQUALITY*

Yang Shangjun (杨尚俊) Cai Qian(蔡 茜)

Abstract *We prove several inequalies for symmetric postive semidefinite, general M-matrices and inverse M-matrices which are generalization of the classical Oppenheim's Inequality for symmetric positive semidefinite matrices.*

Key words *Hadamard's inequality, Fischer's inequality, Oppenheim's inequality, M-matrices, inverse M-matrices, Hadamard product.*

AMS(2000)subject classifications $15A$

For simplicity we denote the set of all $n\times n$ positive semidefinite, symmetric positive semidefinite, nonsingular M-matrices, general M-matrices, inverse M-matrices by $\mathcal{P}, \mathcal{SP}, \mathcal{M}, \overline{\mathcal{M}}, \mathcal{M}^{-1}$, respectively; denote the Hadamard product of A, B by $A\circ B$; dneote the $(n-1)$ th leading principal submatrix of the $n\times n$ matrix A by $A(n)$.

The following inequality is known as Oppenheim's inequality:

Theorem OPP ([2],Theorem 7.8.6)) If $A, B\in\mathcal{SP}$, then

$$(\det A)\prod_{i=1}^{n} b_{ii} = b_{11}\cdots b_{nn} \le \det A\circ B. \tag{1}$$

We shall establish several inequalities which generalize Oppenheims inequality. First we give some lemmas.

Lemma 1 $A, B\in M_n(R)$ satisfy inequality (1) if and only if for arbitrary positive diagonal matrices $D_1, D_2, \hat{A}=D_1A, \hat{B}=BD_2$ satisfy (1).

Proof Suppose that the real matrices A, B satisfy inequality (1). Then

$$\begin{aligned}(\det\hat{A})(\hat{b}_{11}\cdots\hat{b}_{nn}) &= (\det D_1)(\det A)(b_{11}\cdots b_{nn})(\det D_2)\le(\det D_1)(\det A\circ B)(\det D_2)\\ &= \det(D_1A)\circ(BD_2)) = \det\hat{A}\circ\hat{B}\end{aligned}$$

---

* Supported by National Natural Science Foundation of China(60375010).

Received: Dec. 13, 2002.

as desired. Since $A = D_1^{-1}\hat{A}, B = D_2^{-1}\hat{B}$ with D_1^{-1}, D_2^{-1} being positive diagonal, the converse part also holds.

Lemma 2 If $A \in \mathcal{M} \cup \mathcal{M}^{-1}$, then there is a positive diagonal matrix D such that $AD + DA^T \in \mathcal{P}$.

Proof When $A \in \mathcal{M}$, the result is well known (see Theorem 2.5.3. of [3]).

If $A \in \mathcal{M}^{-1}$, then $A^{-1} \in \mathcal{M}$ and for some positive diagonal matrix D we have $A^{-1}D + DA^{-T} \in \mathcal{P}$ from which $DA^T + AD \in \mathcal{P}$ follows.

Lemma 3 For any $n \times n$ real matrix $A, H(A) = A + A^T \in P$ implies $\det A > 0$.

Proof Let $F(A) = \{x * Ax : x \in C^n, x * x = 1\}, \sigma(A)$ be the field of values of A (see chapter 1 of [3]) and the spectrum of A, respectively. Then $\sigma(A) \subset F(A) \subset \{z \in C : \mathrm{Re}(z) > 0\}$ by properties 1.2.5 and 1.2.6 of [3] which imply A is positive stable, then $\det A > 0$ by observation 2.1.4 of [3].

Definition[3] An $n \times n$ real matrix A is strictly row diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i}^{n} |a_{ij}|;$$

A is strictly diagonally dominant of its column entries if $|a_{jj}| > |a_{ij}|, \forall i \neq j$.

Proposition 1 (i) if A is strictly row diagonally dominant, then $\det A > 0$ and A^{-1} is strictly diagonally dominant of its column entries. (ii) if $A \in \mathcal{M}$, then there is a positive diagonal matrix D such that AD is strictly row diagonally dominant. (iii) if $A \in \mathcal{M}^{-1}$, then there exist positive diagonal matrices D_1, D_2 such that $D_1AD_2 = (\alpha_{ij})$ satisfy $\alpha_{ii} = 1, \forall i; \alpha_{ij} < 1, \forall i \neq j$.

Proof (i) and (ii) are known (see Chapter 2 of [3]); and (iii) can be easily deduced from (i) and (ii).

Lemma 4 If $A \in \mathcal{P} \cup \mathcal{M}$ and $B \in \mathcal{P} \cup \mathcal{M} \cup \mathcal{M}^{-1}$, then $\det(A \circ B) > 0$.

Proof If $A, B \in \mathcal{P}$, then $A \circ B \in \mathcal{P}$ by Schur product theorem (Theorem 7.5.3 of [2]), hence $\det(A \circ B) > 0$ as desired.

If $A \in \mathcal{P}, B \in \mathcal{M} \cup \mathcal{M}^{-1}$, then there is a positive diagonal matrix D such that $BD + DB^T \in \mathcal{P}$ by Lemma 2 and $A \circ (BD) + (A \circ (BD))^T = A \circ (BD + DB^T) \in \mathcal{P}$ by Schur product theorem.

Therefore $\det(A \circ (BD)) > 0$ holds by Lemma 3. Now we have

$$\det(A \circ B)\det D = \det((A \circ B)D) = \det(A \circ (BD)) > 0.$$

Since $\det D > 0$, the desired conclusion follows.

If $A \in \mathcal{M}, B \in \mathcal{M} \cup \mathcal{M}^{-1}$, then from Propsotion 1 and Lemma 1 we may assume, without loss of generality, that A is strictly row diagonally dominant and B is strictly row diagonally