INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 2, Pages 293–310

NONCONFORMING MIXED FINITE ELEMENT METHOD FOR THE STATIONARY CONDUCTION-CONVECTION PROBLEM

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(Communicated by Xuecheng Tai)

Abstract. In this paper, a new stable nonconforming mixed finite element scheme is proposed for the stationary conduction-convection problem, in which a new low order Crouzeix-Raviart type nonconforming rectangular element is taken as approximation space for the velocity, the piecewise constant element for the pressure and the bilinear element for the temperature, respectively. The convergence analysis is presented and the optimal error estimates in a broken H^1-norm for the velocity, L^2-norm for the pressure and H^1-seminorm for the temperature are derived.

Key Words. stationary conduction-convection problem, nonconforming mixed finite element, the optimal error estimates.

1. Introduction

We consider the following stationary conduction-convection problem (cf. [1-4]):
Problem (I) Find $u = (u^1, u^2)$, p and T such that

$$(1.1)\qquad \begin{cases} -\mu\triangle u + (u\cdot\nabla)u + \nabla p = \lambda jT, & \text{in } \Omega,\\ \mathrm{div} u = 0, & \text{in } \Omega,\\ -\triangle T + \lambda u\cdot\nabla T = 0, & \text{in } \Omega,\\ u = 0,\quad T = T_0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^2$ is a bounded domain with boundary $\partial\Omega$, u denotes the fluid velocity vector field, p the pressure field, T the temperature field, $\mu > 0$ the coefficient of the kinematic viscosity, $\lambda > 0$ the Groshoff number, $j = (0, 1)$ the two-dimensional vector and T_0 the given initial scale function.

The stationary conduction-convection problem (I) is the coupled equations governing steady viscous incompressible flow and heat transfer process, where incompressible fluid is the Boussinnesq's approximation of the Navier-Stokes equations. In atmospheric dynamics it is an important compelling dissipative nonlinear system, which contains not only the velocity vector field and the pressure field but also the temperature field. From the thermodynamics point of view, we know that the movement of the fluid must have viscosity which will produce quantity of heat. Thus, the movement of the fluid must be companied with mutual transformation of temperature, speed and pressure. Therefore, it is very universal to investigate this nonlinear system.

So far some numerical methods have been studied on the conduction-convection problem (cf. [1-2,5-8]), but much less attention is paid to the theoretical error

Received by the editors June 20, 2007 and, in revised form, October 10, 2008.
2000 *Mathematics Subject Classification.* 65N15, 65N30.
This research was supported by NSF of China (NO.10671184).

analysis of the mixed finite element methods. Shen [9] firstly analyzed the existence and uniqueness of approximation solution, and gave first order accuracy with Bernadi-Raugel element [25] in terms of the small Groshoff number λ that appears in Problem (I) (refer to [10] for the detailed proof). However, the analysis in [9-10] is about the conforming finite elements. Indeed, it seems that there are few studies focusing on the approximation of Problem (I) with the nonconforming finite elements. Recently, these elements have attracted increasing attention from scientists and engineers in various areas since they have some practical advantages. On the one hand, they are usually much easier to be constructed to satisfy the discrete inf-sup condition than the conforming ones, which is usually required in the mixed finite element analysis. On the other hand, from the domain decomposition methods point of view, since the unknowns are associated with the element faces, each degree of freedom belongs to at most two elements, the use of the nonconforming finite elements facilitates the exchange of information across each subdomain and provides spectral radius estimates for the iterative domain decomposition operator [23].

The main aim of this paper is to construct a new low order Crouzeix-Raviart type nonconforming rectangular element and apply it to Problem (I) with a mixed finite element scheme. By virtue of the element's special properties, the convergence analysis is presented and the optimal error estimates are obtained. The remainder of this paper is organized as follows. In section 2, we introduce the variational formulation of Problem (I) and briefly recite the existence and uniqueness of its solution proved in [9-10]. In section 3, we first give the construction of Crouzeix-Raviart type nonconforming mixed finite element scheme and then prove that the approximation spaces of the velocity and the pressure satisfy the discrete inf-sup condition, which yields the existence and uniqueness of approximation solution. In the last section, some important lemmas, the convergence analysis and the optimal error estimates are obtained.

We will employ standard definitions for the Sobolev spaces $W^{k,p}(\Omega)$ with norm $\|\cdot\|_{k,p,\Omega}$, and $H^k(\Omega) = W^{k,2}(\Omega)$ with norm $\|\cdot\|_k$, respectively (cf. [17]). Throughout the paper, C indicates a positive constant, possibly different at different occurrences, which is independent of the mesh parameter h, but may depend on Ω and other parameters introduced in this paper. Notations not especially explained are used with their usual meanings.

2. The existence and uniqueness of the solution to the variational formulation

The variational formulation of Problem (I) is written as:
Problem (I^*) Find $(u, p, T) \in X \times M \times W$, such that $T|_{\partial\Omega} = T_0$

$$\left\{ \begin{array}{ll} a(u,v) + a_1(u;u,v) - b(p,v) = \lambda(jT,v), & \forall\, v \in X, \\ b(q,u) = 0, & \forall\, q \in M, \\ d(T,\varphi) + \lambda \bar{a}_1(u;T,\varphi) = 0, & \forall\, \varphi \in W_0, \end{array} \right. \tag{2.1}$$

where

$$X = H_0^1(\Omega)^2, M = L_0^2(\Omega) = \left\{ q, q \in L^2(\Omega), \int_\Omega q dx = 0 \right\}, W = H^1(\Omega), W_0 = H_0^1(\Omega),$$