J. Partial Differential Equations
Vol. 2, *No.* 4(1989), 22—34

THE CENTER MANIFOLD REDUCTION METHOD FOR HOPF BIFURCATION THEOREM

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(Received October 7, 1987; Revised January 11, 1989)

Abstract

In this paper we apply the center manifold reduction method to prove a Hopf bifurcation theorem for infinite dimensional problem. The asymptotic expression of bifurcation formulae and stability condition are given. The Hopf bifurcation problem for a system of parabolic equations is considered.

1. Introduction

Hopf E. published his famous paper [1] about the bifurcation periodic solution for ordinary differential equation in 1942, since then a substantial literature on generalizations and related problems have been developed. In recent years, some work on this problem has gone into proving the Hopf bifurcation theorem in the context of abstract evolution equation. In paper [2] Crandall and Rabinowitz used the Lyapunov-Schmidt method to prove the Hopf bifurcation theorem for a class of infinite dimensional problem. Iooss obtained in [3] an existence result of periodic solution bifurcation for a class of infinite dimensional problem in Hilbert space. Henry D. [4] treated an abstract evolution equation in a Banach space using an infinite dimensional version of the center manifold theorem. He required some conditions similar to [2].

In this paper we use the center manifold reduction method to prove the Hopf bifurcation theorem for a class of abstract evolution equations in Banach space. Our conditions are different from the conditions in [2, 4]. Our results can be applied to partial functional differential equations, and the result in [2] can not be applied directly. We shall give this result elsewhere. Moreover, we use some results in [5] to give the bifurcation formulae, the asymptotic expression of the bifurcation periodic solution and the stability results.

2. Relative Results for Center Manifolds

Let Z be a Banach space with norm $\| \cdot \|$. We shall consider an abstract evolution equation on Z of the form:

$$\frac{du}{dt} = Cu + f(u) \tag{2.1}$$

where C is the generator of a strongly continuous semigroup $S(t)$ and $f: Z \to Z$ is smooth enough with $f(0)=0, Df(0)=0$ (D denotes the Frechét derivate); Moreover, we assume that:

(i) $Z = X \oplus Y$, where X is finite dimensional subspace of Z, Y is infinite dimensional closed subspace of Z.

(ii) X is invariant under the action of C (called C-invariant). Let A be the restriction of C to X, we assume that the real parts of the eigenvalue of A are all zero.

(iii) Let $U(t)$ be the restriction of $S(t)$ to Y, assume that Y is $U(t)$-invariant for each $t \geq 0$ and that:

$$\|U(t)\| \leq ae^{-bt} \quad \forall\ t \geq 0$$

where a, b are positive constants.

Let P be the normal projection operator of Z to X along Y, $B = (I-P)C$, $f(x,y) = Pf(x+y)$, $g(x,y) = (I-P)f(x+y)$, then equation (2.1) takes the form

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x,y) \\ \frac{dy}{dt} &= By + g(x,y) \end{aligned} \tag{2.2}$$

Definition 1 *A set $M \subset Z$ is an invariant manifold for (2.2) if for any solution $(x(t), y(t))$ of (2.2) with $(x(0), y(0)) \in M$, implies that for some $T > 0$, $(x(t), y(t)) \in M$ for all $t \in [0, T]$.*

Definition 2 *An invariant manifold $M = \{(x, h(x)) \mid \|x\| < \delta\}$ for (2.2) is a center manifold if $h(0) = 0, Dh(0) = 0$.*

The existence result for center manifold in [6] is:

Theorem 2.1 *suppose that conditions (i)—(iii) are satisfied, let $f(u) \in C^k (k \geq 2)$, and assume that $f^{(k)}(u)$ is uniformly continuous. Then there exists a center manifold $M = \{(x, h(x)) \mid \|x\| < \delta\}$ for (2.2), where $h(x) \in C^k$.*

Remark 1 Since the proof method of center manifolds' existence is a local method (see[7]), we can also obtain Theorem 2.1 with $h(x) \in C^{k-1}$ without the condition "$f(u)$ is uniformly continuous".

The flow governed by equation (2.2) on the center manifold may be represented