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LOCAL SOLVABILITY FOR A CLASS OF NONHOMOGENEOUS LEFT INVARIANT DIFFERENTIAL OPERATORS ON $H_n \otimes R^K$

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Abstract In this paper we discuss the local solvability of the following nonhomogeneous left invariant differential operators on the nilpotent Lie group $H_n \otimes R^K$:

$$P(X,Y,T,Z) = \sum_{\substack{|\alpha+\beta|+l+|\gamma|\leqslant m \\ |\alpha+\beta|+2l=s}} a_{\alpha\beta l\gamma} X^\alpha Y^\beta T^l Z^\gamma$$

where $X_j, Y_j (j=1,2,\cdots,n), T, Z_j (j=1,2,\cdots,K)$ are bases of left invariant vector fields on $H_n \otimes R^K$ and $a_{\alpha\beta l\gamma}$ are complex constants.

Key Words Local solvability; nilpotent Lie group; nonhomogeneous left invariant differential operator.

Classifications 35A05; 35A07.

The nilpotent Lie group $H_n \otimes R^K$ is another important noncommutative nilpotent Lie group except for the Heisenberg group H_n. The study of Cauchy problems, Dirichlet problems on H_n ([1], [2]) and some problems for functions with several complex variables all need to use these groups. The model group in the study of $\bar{\partial}_b$ complex on *CR* manifolds with degenerate Levi form is not H_n but $H_n \otimes R^K$[3]. Because of the difference of structures between $H_n \otimes R^K$ and H_n, the discussions of many problems on $H_n \otimes R^K$ are more difficult than those on H_n.

In this paper we discuss the local solvability of the left invariant differential operators on $H_n \otimes R^K$ with the form

$$P(X,Y,T,Z) = \sum_{\substack{|\alpha+\beta|+l+|\gamma|\leqslant m \\ |\alpha+\beta|+2l=s}} a_{\alpha\beta l\gamma} X^\alpha Y^\beta T^l Z^\gamma \tag{1}$$

where $a_{\alpha\beta l\gamma}$ are complex constants.

So far, there is very little investigation of local solvability of nonhomogeneous operators on general nilpotent Lie groups even on the Heisenberg group, because this is a very difficult problem. The operators (1) are a special class of nonhomogeneous operators on $H_n \otimes R^K$. The result obtained here gives an important improvement of the result in [4].

Let $(x,y,t,z) \in R^n \times R^n \times R^1 \times R^K$ and the nilpotent Lie group $H_n \otimes R^K$ consist of the elements (x,y,t,z) with the multiplication

$$(x,y,t,z)(x',y',t',z') = (x+x',y+y',t+t'+\frac{1}{2}\sum_{j=1}^{n}(x_jy'_j - y_jx'_j),z+z')$$

The Lie algebra $\mathscr{G}$ of $H_n \otimes R^K$, which is equivalent to the left invariant vector fields on $H_n \otimes R^K$, has the basis

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2}y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}x_j\frac{\partial}{\partial t}, \quad j=1,2,\cdots,n$$

$$T = \frac{\partial}{\partial t}, \quad Z_j = \frac{\partial}{\partial z_j}, \quad j=1,\cdots,K$$

The commutative relations are given by $[X_i, Y_j] = \delta_{ij}T$ and all the others are zero. There are two equivalent classes of irreducible unitary representations on $H_n \otimes R^K$[5]. The first is the L^2 representations given by

$$\Pi_{\lambda\mu}(x,y,t,z)f(\eta) = \exp\{i(\lambda t + \mu\cdot z + (\mathrm{sgn}\lambda)|\lambda|^{\frac{1}{2}}y\cdot\eta + \frac{1}{2}\lambda x\cdot y)\}$$

$$\cdot f(\eta + |\lambda|^{\frac{1}{2}}x), \quad \text{for all } f(\eta)\in L^2(R^n) \tag{2}$$

where $(\lambda,\mu)\in(R^1\backslash 0)\times R^K$ are parameters, whose induced representations on $\mathscr{G}$ are

$$\Pi_{\lambda\mu}(X_j) = |\lambda|^{\frac{1}{2}}\frac{\partial}{\partial\eta_j}, \quad \Pi_{\lambda\mu}(Y_j) = i(\mathrm{sgn}\lambda)|\lambda|^{\frac{1}{2}}\eta_j, \quad j=1,2,\cdots,n$$

$$\Pi_{\lambda\mu}(T) = i\lambda, \quad \Pi_{\lambda\mu}(Z_j) = i\mu_j, \quad j=1,2,\cdots,K$$

The second is

$$\Pi_{abc}(x,y,t,z) = \exp\{i(a\cdot x + b\cdot y + c\cdot z)\}$$

where $(a,b,c)\in R^n\times R^n\times R^K$ are parameters, whose induced representations on $\mathscr{G}$ are given by

$$\Pi_{abc}(X_j) = ia_j, \quad \Pi_{abc}(Y_j) = ib_j, \quad j=1,2,\cdots,n$$

$$\Pi_{abc}(T) = 0, \quad \Pi_{abc}(Z_j) = ic_j, \quad j=1,2,\cdots,K$$

$H_n \otimes R^K$ has a dilation $\delta_r(x,y,t,z) = (rx,ry,r^2t,rz)$ and the induced dilation on $\mathscr{G}$ is $\delta_r(X_j) = rX_j, \delta_r(Y_j) = rY_j, \delta_r(T) = r^2T, \delta_r(Z_j) = rZ_j$. A left invariant differential operator P is called homogeneous with degree m if $\delta_r(P) = r^mP$. Hence the operator (1) is nonhomogeneous. The Plancherel formula on $H_n \otimes R^K$ is[4]

$$\int_{H_n\otimes R^K} |f(x,y,t,z)|^2 dxdydtdz = (2\pi)^{-(n+K+1)}\int_{R^1\backslash 0}\int_{R^K} \|\Pi_{\lambda\mu}f\|_{HS}^2 |\lambda|^n d\lambda d\mu \tag{3}$$