J. Partial Differential Equations
Vol. 4, *No.* 4 (1991), 51–65

TWO PARABOLIC EQUATIONS WITH HYSTERESIS

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(Received July 28, 1989; revised March 5, 1990)

Abstract In this paper we discuss the properties of hysteresis phenomena and give some conclusions about two parabolic equations with hysteresis raised by A. Visintin.
Key Words Hysteresis operator; Leray-Schauder fixed point theorem; periodic solution; parabolic equation.
Classifications 35A05; 35K55.

The evolution equations containing hysteresis factor (denoted by $\mathcal{F}$ below) aroused much interest in recent years. The following two equations

$$\frac{\partial \mathcal{F}(u)}{\partial t} + Au = f \tag{0.1}$$

$$\frac{\partial u}{\partial t} + Au + \mathcal{F} = f \tag{0.2}$$

(where A is a symmetric and uniform elliptic operator) were raised and given existence of weak solution for initial-boundary value problem by A.Visintin in [1]. In this paper we shall first deal with properties of hysteresis operator and then proceed to do the existence and uniqueness of the solution for initial-boundary value problem and asymptotic behavior of this solution to (0.2), and the existence of the periodic solution for boundary value problem to (0.1) and (0.2).

1. Hystere Operator

Such a mapping often occurs that $\forall v : [0, T] \to R$, $\forall z(0) \in R$, there exists a unique $z : [0, T] \to R$ such that the value $z(t)$ depends on the structure of v in $[0, T]$ and $z(0)$. Denoting by $\mathcal{F}$ this mapping, we have

$$z(t) = \mathcal{F}(v(\,\cdot\,)|_{[0,t]},\ z(0)) \quad \text{for } 0 < t \leq T \tag{1.1}$$

If $\mathcal{F}$ satisfies the following (1.2)–(1.6), then $\mathcal{F}$ is called a hysteresis operator on $C^0([0,T])$:

There exists

$$S : R \times [0,T] \to \mathcal{T}(R) \tag{1.2}$$

$$\begin{cases} \mathrm{Dom}(\mathcal{F}) = \{(v,t,\xi) | v \in C^0([0,T]),\ 0 \le t \le T,\ \xi \in S(v(0),0)\} \\ \forall (v,t,\xi) \in \mathrm{Dom}(\mathcal{F}),\ \mathcal{F}(v,t,\xi) \in S(v(t),t) \end{cases} \tag{1.3}$$

$$\begin{cases} \forall v \in C^0([0,T]),\ \forall \xi \in S(v(0),0),\ t \mapsto \mathcal{F}(v,t,\xi) \text{ is} \\ \text{continuous in } [0,T] \end{cases} \tag{1.4}$$

$$\forall v \in C^0([0,T]),\ \forall \xi \in S(v(0),0),\ \mathcal{F}(v,0,\xi) = \xi \tag{1.5}$$

$$\begin{cases} \forall \bar{t} \in [0,T],\ \forall v_1, v_2 \in C^0([0,T]),\ \text{if } v_1 = v_2 \text{ in } [0,\bar{t}] \\ \text{then } \forall \xi \in S(v_1(0),0),\ \mathcal{F}(v_1,\bar{t},\xi) = \mathcal{F}(v_2,\bar{t},\xi) \end{cases} \tag{1.6}$$

hence, (1.1) can be rewritten as

$$z(t) = \mathcal{F}(v,t,z(0)) \quad \text{for } 0 \le t \le T,\ z(0) = \xi \tag{1.7}$$

Assume in this paper, moreover, that

$$\begin{cases} \forall [t',t''] \subset [0,T],\ \forall v \in C^0([0,T]),\ \text{if } v(t) = \text{const.}\ \forall t \in [t',t''] \\ \text{then } \forall \xi \in S(v(0),0),\ \mathcal{F}(v,t,\xi) = \text{const.}\ \forall t \in [t',t''] \end{cases} \tag{1.8}$$

and the choice of $\xi \in S(v(0),0)$ depends uniquely on $v(0)$, hence (1.7) can be rewritten as

$$z(t) = \mathcal{F}(v)(t) \quad \text{for } 0 \le t \le T \tag{1.9}$$

Example There is excessive SO_2 with density $v(x,t)$ in the air over an area D. A protector M in D can absorb SO_2 in the case of over-density while release a bit of SO_2 when the density is very low. Due to the limited sensitivity, the absorbability $z(t)$ of M to $v(t)$ is generally subject to $z(t) = \mathcal{F}(v)(t)$. If $f(x,t)$ stands for the source in D causing excessive SO_2, then $v(x,t)$ suits equation (0.2). $\mathcal{F}$ can be defined as follows:

Take constants $p > q > 0$, $\beta > \gamma > 0$, and set an odd function $g(v) \in C^1([-p,p])$ and $\gamma \le g'(v) < \beta$ (see Figure). Let l_1, l_2 be $z = \beta(v+q)$ and $z = \beta(v-q)$ respectively.