J. Partial Differential Equations
Vol. 5, *No.* 1 (1992), 57–68

EVERYWHERE REGULARITY FOR WEAK SOLUTIONS TO VARIATIONAL INEQUALITIES OF TRIANGULAR FORM

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(Received Jan. 7, 1989; revised July 21, 1990)

Abstract In this paper, we obtain two results of weak solutions to variational inequalities of triangular form under controllable growth and a class of natural growth conditions, i.e. 1°. L^p-estimate for the gradient; 2°. $C^{1,\beta}_{\text{loc}}(\Omega, R^N)$ regularity.
Key Words Regularity; variational inequality of triangular form; diagonal elliptic condition.
Classification 35J.

1. Introduction

Lewy[1], Lewy-Stampacchia[2], Brezis-Stampacchia[3] and Giaquinta M.[4] et al. have made a systematic study on the convex obstacle problem. But for vector-valued functions, especially for the case $n \geq 3$, there are a few papers dealing with the regularity. Under the quadratic growth condition, Hildebrandt S. and Widman K-O.[5] have proved the regularity to variational inequalities of diagonal form with general obstacle. This paper proves the regularities for variational inequalities of triangular form with some special obstacle. We emphasize that the diagonal ellipticity condition we introduced here is weaker than the strong ellipticity condition and under natural growth condition (I), we seek solutions in $\mathcal{C} \cap L^{n(r-1)/(2-r)}(\Omega, R^N)$ which is larger than $\mathcal{C} \cap L^\infty(\Omega, R^N)$.

We finding $u \in \mathcal{C}$ satisfying the variational inequality

$$\int_\Omega [A^{\alpha\beta}_{ij}(x,u)D_\beta u^j + a_i(x,u)]D_\alpha(u^i - v^i)dx$$
$$\leq \int_\Omega B_i(x,u,Du)(u^i - v^i)dx, \quad \forall v \in \mathcal{C} \tag{1.1}$$

where $\mathcal{C} = \{v \in H^1(\Omega, R^N) : v^k \geq \psi^k,\ k = 1,2,\cdots,N \text{ in } \Omega \subset R^n, n \geq 3;\ v - v_0 \in H^1_0(\Omega, R^N)\}$, is a closed convex set, v_0 and ψ are prescribed functions with $\psi^k|_{\partial\Omega} \leq v_0^k|_{\partial\Omega}$

We assume that

(i) $A^{\alpha\beta}_{ij}(x,u) = 0$, when $j > i$, (Triangular condition)

$$A_{kk}^{\alpha\beta}(x,u)\xi_\alpha\xi_\beta \geq \lambda|\xi|^2,\ \forall\xi \in R^n,\ \lambda > 0;\ \text{(Diagonal elliptic condition)}$$

(ii) Controllable growth conditions:

$$\begin{aligned} &|a_k^\alpha(x,u)| \leq C(|u|^{n/(n-2)} + f_k^\alpha),\ f_k^\alpha \in L^\sigma(\Omega), \sigma > n \\ &|B_k(x,u,p)| \leq C(|p|^{(n+2)/n} + |u|^{(n+2)/(n-2)} + g_k),\ g_k \in L^s(\Omega), s > n/2 \end{aligned} \tag{1.2}$$

$(\text{iii})_1$ Natural growth conditions:

$$u \in \mathcal{C} \cap L^{n(r-1)/(2-r)}(\Omega, R^N) \text{and satisfies (2) and}$$

$$|B_k(x,u,p)| \leq C(|\ p\ |^r + |\ u\ |^{(n+2)/(n-2)} + g_k)$$

$$g_k \in L^s(\Omega), s > n/2, (1 + 2/n) < r < 2$$

or $(\text{iii})_2$ $u \in \mathcal{C} \cap L^\infty(\Omega, R^N)$. $M = \sup_\Omega |u|$ and satisfy $a_k^\alpha(x,u) \in L^\sigma(\Omega \times R^N), \sigma \geq 3$,

$$|B_k(x,u,p)| \leq a\sum_{j=1}^k |p_j|^2 + b_k(x), b_k \in L^s(\Omega), s \geq 2,\ \text{a} - \text{const.}$$

The repeated indices i, j are to be summed from 1 to N, while the repeated indices α, β are to be summed from 1 to n; but the repeated index k doesn't. C represents the constant at various cases. $2^* = 2n/(n-2), q = 2n/(n+2), Du = \{D_\alpha u^i : \alpha = 1,2,\cdots,n; i = 1,2,\cdots,N\}, u_R^k = \fint_{B_R} u^k dx = \frac{1}{|B_R|}\int_{B_R} u^k dx.$

2. L^p-estimates

Proposition 2.1 *Suppose that* (i) *and* (ii) *are satisfied and* $\psi \in H^{1,\sigma}(\Omega, R^N), \sigma > n$, *if* u *is a weak solution to* (1), *then there exists a constant* $p > 2$, *such that* $u \in H_{\text{loc}}^{1,p}(\Omega, R^N)$, *and for any* $x_0 \in \Omega$ *and* $R_0 < \text{dist}(x_0, \partial\Omega)$, *when* $R < R_0$, *the following estimate holds*

$$\begin{aligned} \Big(\fint_{B_{R/2}} (|Du|^2 + |u|^{2^*})^{p/2} dx\Big)^{1/p} &\leq C\Big\{\Big(\fint_{B_R} (|Du|^2 + |u|^{2^*}) dx\Big)^{1/2} \\ &+ \Big(\fint_{B_R} (\sum_{\alpha,i} |f_i|^2 + |D\psi|^2)^{p/2} dx\Big)^{1/p} + R\Big(\fint_{B_R} \sum_i |g_i|^{pq/2} dx\Big)^{2/pq}\Big\} \end{aligned} \tag{2.1}$$

Proof Take $\eta \in C_0^\infty(B_R), 0 \leq \eta \leq 1, \eta \equiv 1$ in $B_{R/2}, |D\eta| \leq C/R$. Since $u^k \geq \psi^k$, then $u_R^k \geq \psi_R^k$. Setting $v^k = \psi^k + (1-\eta^2)(u^k - \psi^k) + \eta^2(u_R^k - \psi_R^k), v^i = u^i, i \neq k$, and