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INITIAL VALUE PROBLEM FOR A NONLINEAR EVOLUTION SYSTEM WITH SINGULAR INTEGRAL DIFFERENTIAL TERMS

Zhang Linghai
(Institute of Applied Physics and Computational Mathematics, Beijing 100088)
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Abstract The initial value problem for a nonlinear evolution system with singular integral differential terms is studied. By means of *a priori* estimates of the solutions and Leray-Schauder's fixed point theorem, we demonstrate the existence and uniqueness theorems of the generalized and classical global solutions to the problem.

Key Words Initial value problem; integral estimate; nonlinear evolution system; singular integral differential term.

Classification 35Q20.

1. Introduction

In this paper, we study the initial value problem (IVP) for the following nonlinear evolution system with singular integral differential terms (NES with SIDT) [5–7]

$$U_t + U_{x^{2p+1}} + [\text{grad } \Phi(U)]_x + \alpha H U_{x^{2r}} + (-1)^s \beta H U_{x^{2s-1}} + \gamma H U = A(x,t)U + g(x,t) \tag{1}$$

$$U(x,0) = U_0(x) \tag{2}$$

in the unbounded domain $Q_T = \{(x,t) : -\infty < x < \infty,\ 0 \le t \le T\}$, where H is the Hilbert singular integral operator

$$HU(x,t) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{U(y,t)}{y-x} dy \tag{3}$$

In system (1), $U(x,t) = (U_1(x,t), \cdots, U_N(x,t))$ is a N-dimensional vector valued unknown function of the two real variables $-\infty < x < \infty$ and $t \ge 0$; $\Phi(U)$ is a scalar function of the vector variable U; "grad" denotes the gradient operator with respect to the vector variable U; $A(x,t)$ is a $N \times N$ matrix of functions $a_{i,j}(x,t) (1 \le i,j \le N)$, $g(x,t)$ is a N-dimensional vector valued function of functions $g_i(x,t) (1 \le i \le N)$; α, β, and γ are real constants; $p \ge 1$, $1 \le r, s \le p$ are integers.

System (1) is a much generalized NES. In fact, if $\alpha = \beta = \gamma = 0$, then it is the generalized Korteweg-de Vries (KdV) system of higher order [1]

$$U_t + U_{x^{2p+1}} + [\text{grad } \Phi(U)]_x = A(x,t)U + g(x,t) \tag{4}$$

In order to study the IVP (2) for the NES with SIDT (1), we need to investigate the IVP (2) for the corresponding NES with dissipative term

$$U_t + (-1)^{p+1}\varepsilon U_{x^{2p+2}} + U_{x^{2p+1}} + [\text{grad } \Phi(U)]_x + \alpha H U_{x^{2r}}$$
$$+ (-1)^s \beta H U_{x^{2s-1}} + \gamma H U = A(x,t)U + g(x,t) \tag{5}$$

where $0 < \varepsilon < 1$. The solution of problem (1,2) will be obtained by the limiting procedure of approaching to zero of the dissipative coefficient ε for the solution of problem (5, 2).

If $f(x) \in L^1(\boldsymbol{R}) \cap L^2(\boldsymbol{R})$, define its Fourier transform as follows:

$$F[f](\zeta) \equiv \hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x) \exp(-ix\zeta) dx \tag{6}$$

Let the $L^p (1 \le p \le \infty)$ norm on $\boldsymbol{R}$ be denoted by $\|U\|_{L^p}$, and define the Sobolev spaces H^m and H_0^m by means of the norms

$$\|U\|_{H^m} = \Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} (1+|\zeta|^2)^m |\hat{U}(\zeta)|^2 d\zeta\Big]^{1/2} \tag{7}$$

$$\|U\|_{H_0^m} = \Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} |\zeta|^{2m} |\hat{U}(\zeta)|^2 d\zeta\Big]^{1/2} \tag{8}$$

where $m \ge 0$.

For simplicity, we will denote by C any positive constant appeared in our paper, which depends only on the coefficients α, β, and γ, the norms of the initial function $U_0(x)$, and the norms of $A(x,t)$ and $g(x,t)$. Furthermore, we regard

$$\|U(t)\| = \|U(t)\|_{L^2(\boldsymbol{R})}, \ \|U(t)\|_\infty = \|U(t)\|_{L^\infty(\boldsymbol{R})}$$
$$\|U(t)\|_m = \|U(t)\|_{H^m(\boldsymbol{R})}, \ |U(t)|_m = \|U(t)\|_{H_0^m(\boldsymbol{R})}$$

Now let us introduce some functional spaces.

$$B = L^\infty(0,T; H^1(\boldsymbol{R})), \ \overline{B} = \{U = (U_1, \cdots, U_N) \in \boldsymbol{R}^N : U_i \in B, \ 1 \le i \le N\}$$
$$Z = L^\infty(0,T; H^{p+1}(\boldsymbol{R})) \cap L^2(0,T; H^{2p+2}(\boldsymbol{R})) \cap H^1(0,T; L^2(\boldsymbol{R}))$$
$$\overline{Z} = \{U = (U_1, \cdots, U_N) \in \boldsymbol{R}^N : U_i \in Z, \ 1 \le i \le N\}$$

If $U(x,t) \in B$ (or Z), define its norm as follows:

$$\|U\|_B^2 = \sup_{0 \le t \le T} \|U(t)\|_1^2$$

$$\|U\|_Z^2 = \sup_{0 \le t \le T} \|U(t)\|_{p+1}^2 + \int_0^T \|U(t)\|_{2p+2}^2 dt + \int_0^T \|U_t(t)\|^2 dt$$