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THE GLOBAL SOLUTION OF THE SCALAR NONCONVEX CONSERVATION LAW WITH BOUNDARY CONDITION*

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Abstract Using the polygonal approximations method, we construct the global approximate solution of the initial boundary value problem (1.1)–(1.3) for the scalar nonconvex conservation law, and prove its convergence. The crux of this work is to clarify the behavior of the approximations on the boundary $x = 0$.
Key Words Scalar conservation law; nonconvex; boundary condition; polygonal approximations method.
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1. Introduction

From the 1960's to the early 1980's, the global solution of initial value problem for the scalar conservation law without convexity condition was investigated by various methods[1–7]. Using various methods to study the solution, we do not mean to improve the methods, but, show that different methods have different meanings.

The problem with boundary condition, namely, the initial boundary value problem for the scalar conservation law was first discussed in 1979 by Bardos, Leroux and Nedelec[8]. They showed the existence of the global weak solution by vanishing viscosity method. Of cause, the main difficulty is to clarify the behavior of the solution on the boundary, because the equation is nonlinear. It is well-known that the existence is only a beginning of study, but unterminus. To obtain more understanding for the solution, in 1988, Le Floch[9] derived explicit formula for the exact solution of the initial boundary value problem (in the quarter plane). For the same problem, in 1991, Joseph and Gowda[10] derived another kind of explicit formula by a different method. Of cause, the conservation law which they discussed is strictly convex. In the nonconvex case, even for the problem without boundary condition (only with initial condition), the explicit formula of the exact solution has not yet been found up to now.

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In 1972, using the polygonal approximations method, Dafermos[5] constructed the global approximate solution of the initial value problem for scalar conservation law without convexity condition. That approximations possess clear structure, and may be used to take numerical computation. In this paper, by Dafermos' polygonal approximations method, we construct the global approximations of the initial boundary value problem for the scalar nonconvex conservation law

$$\begin{cases} u_t + f(u)_x = 0 & (0 < x < +\infty, t > 0) \quad (1.1) \\ u(0,t) = u_b(t) & (t \geq 0) \quad (1.2) \\ u(x,0) = u_0(x) & (0 \leq x < +\infty) \quad (1.3) \end{cases}$$

where $f(u)$ is a locally Lipschitz continuous function, $u_0(x), u_b(t)$ are bounded and locally bounded variation functions on $[0,+\infty)$. And we also show the convergence of the approximations. The crux of this work is to clarify the behavior of the approximations on the boundary $x = 0$. In addition, following [11] and [12], we show that the approximations in $[0,+\infty) \times [0,+\infty)$ can be constructed in a finite number of step, which is left by Dafermos in [5] (for the initial value problem). Hence the approximations can be used to take numerical computation.

2. Definition of Weak Solution

Following [2], [8] and [13], we give the definition of weak solution to the initial boundary value problem (1.1)–(1.3) as follows:

Definition 2.1 *A locally bounded and measurable function $u(x,t)$ on $[0,+\infty) \times [0,+\infty)$ is called a weak solution of the initial boundary problem (1.1)–(1.3), if for every $k \in R^1$ and for any nonnegative function $\varphi(x,t) \in C_0^\infty([0,+\infty) \times [0,+\infty))$, it satisfies the following inequality:*

$$\int_0^{+\infty}\int_0^{+\infty} \{|u-k|\varphi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\varphi_x\}dxdt + \int_0^{+\infty} |u_0(x)-k|\varphi(x,0)dx$$
$$+ \int_0^{+\infty} \operatorname{sgn}(u_b(t)-k)(f(u(0,t)-f(k))\varphi(0,t)dt \geq 0 \qquad (2.1)$$

where $\operatorname{sgn} x = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

From the following two Lemmas, we can see that Definition 2.1 is well posed.

Lemma 2.1 *If $u(x,t)$ is a weak solution of the (1.1)–(1.3), then $u(0,t) = u_b(t)$ or*

$$\frac{f(u(0,t)) - f(k)}{u(0,t) - k} \leq 0, \quad k \in I(u(0,t), u_b(t)), \ k \neq u(0,t), \text{ a.e. } t \geq 0 \qquad (2.2)$$

where $I(u(0,t), u_b(t)) = [\min\{u(0,t), u_b(t)\}, \max\{u(0,t), u_b(t)\}]$