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THE HOPF BIFURCATION IN A PARABOLIC FREE BOUNDARY PROBLEM WITH DOUBLE LAYERS*

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Abstract We consider a parabolic free boundary problem which has a bifurcation parameter and double interfaces. We investigate the sign change in a real part of eigenvalues and the transversality condition as a bifurcation parameter cross the critical value in order to examine the stability of the stationary solutions. The occurence of a Hopf bifurcation will be shown at a critical value.

Key Words Evolution equation; free boundary problem; parabolic equation; Hopf bifurcation.
Classification 35R35, 35B32, 35B25, 35K22, 35K57, 58F14, 58F22.

1. Introduction

We consider a reaction-diffusion system for a pair of functions (u, v) in which the first component u reacts much faster than the second component v, while u diffuses slower than v:

$$\begin{cases} \varepsilon\tau u_t = \varepsilon^2 u_{xx} + f(u,v) \\ v_t = Dv_{xx} + g(u,v) \end{cases} \tag{1}$$

The above system is assumed to satisfy zero flux boundary conditions at the boundaries.

We are interested in the sigular limit $\varepsilon \downarrow 0$ of a system of the form (1) and assume that the system (1) has a steady state with a double layer on a finite interval. In this case, an analysis of the layer solutions suggests that the layer of width $O(\varepsilon)$ converges to interfacial curves $x = s(t)$ and $x = m(t)$ in x, t-space as $\varepsilon \downarrow 0$ (see [1]). An analysis of the dynamics of this process has been shown (see for example [2], [3], [4]) to lead a free boundary problem consisting of an initial-boundary value problem related to Equation (1).

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In this paper, we are dealing with the following parabolic free boundary problem with double layer

$$\begin{cases} v_t = Dv_{xx} - c^2 v + H(x - s(t)) - H(x - m(t)) & \text{for } (x,t) \in \Omega^- \cup \Omega^+ \\ v_x(0,t) = 0 = v_x(1,t) & \text{for } t > 0 \\ v(x,0) = v_0(x) & \text{for } 0 \le x \le 1 \\ \tau \dfrac{ds}{dt} = C(v(s(t),t)) & \text{for } t > 0 \\ \tau \dfrac{dm}{dt} = -C(v(m(t),t)) & \text{for } t > 0 \\ s(0) = s_0 & \\ m(0) = m_0 & \end{cases} \tag{2}$$

where $v(x,t)$ and $v_x(x,t)$ are assumed to be continuous in Ω. Here $H(y)$ is the Heaviside function, $\Omega = (0,1) \times (0,\infty)$, $\Omega^- = \{(x,t) \in \Omega : 0 < x < s(t), m(t) < x < 1\}$ and $\Omega^+ = \{(x,t) \in \Omega : s(t) < x < m(t)\}$.

The free boundary problem (2) comes from the problem (1) where the reaction terms f and g are of the type investigated by McKean [5], namely

$$f(u,v) = H(u-a) - u - v, \quad g(u,v) = u - \gamma v$$

where $H(y)$ is the Heaviside function. The velocity of the interface $C(v)$ can be calculated explicitly as (see [2], [4])

$$C(v) = \frac{2(v+a)-1}{\sqrt{(1-v-a)(v+a)}}$$

In the following figures, we give the results of some numerical experiments which simulate the evolution of the free boundaries $s(t)$ and $m(t)$. For the purposes of these experiments we have used the parameters $c^2 = 2$, $a = \frac{1}{4}$. For these parameters, the problem (2) has a stationary solution $(v^*(x), s^*, m^*)$ with $s^* = \frac{1}{4}$ and $m^* = \frac{3}{4}$ for all τ illustrated by the dashed line in Figures 1, 2. In each simulation the initial function v_0 was taken to be $v_0(x) = \left(x - \frac{1}{4}\right)\left(x - \frac{3}{4}\right) + \frac{1}{4}$.

In Figure 1, the free boundaries undergo a damped oscillation about the equilibrium values $\left(\frac{1}{4}, \frac{3}{4}\right)$. In Figure 2a, out-of-phase (or, so called antisymmetric) oscillations appear and in Figure 2b, the interfacial curves spiral outward (in the phase plane) toward periodic curves. The initial values $s_0 = 0.15$ and $m_0 = 0.85$ was used in all figures. The results of numerical approximations suggest the following picture of the behavior of solutions in this example. The equilibrium solution with $s(t) = \frac{1}{4}$ and $m(t) = \frac{3}{4}$ is stable for large τ and solutions to (2) tend to this equilibrium as $t \to \infty$.