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ON A MAXIMUM PRINCIPLE AND UNIQUENESS THEOREMS FOR A SYSTEM OF FIRST ORDER EQUATIONS OF MIXED TYPE*

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Abstract A maximum principle for a system of first order equations of mixed type is established. The uniqueness theorems of solutions to the generalized Tricomi type problem and to the Frankl's problem are proved by the method of auxiliary functions.

Key Words System of equations of mixed type; generalized Tricomi type problem; Frankl's problem; auxiliary function; maximum principle.

Classification 35M05.

1. Auxiliary Function and Maximum Principle

In domain $\mathcal{D} \subset R^2$ consider a system of first order equations

$$\begin{cases} Cu_x - Bu_y - v_y + \alpha u + \beta v = f \\ Au_y - Bu_x + v_x + \gamma u + \delta v = g \end{cases} \tag{1}$$

where the coefficient determinant $\Delta \equiv B^2 - AC$ is just the discriminant of type of the system (1), the system (1) is elliptic when $\Delta < 0$, and is hyperbolic when $\Delta > 0$. If $\Delta < 0$ in a subdomain $\mathcal{D}_+$ of $\mathcal{D}$ and $\Delta > 0$ in another subdomain $\mathcal{D}_-(\equiv \mathcal{D}\backslash\mathcal{D}_+)$ of $\mathcal{D}$, then the system (1) in $\mathcal{D}$ is of mixed type. In [1] we considered a special case of (1), i.e., $\beta_x + \delta_y = 0$ in $\mathcal{D}$, in this case the system was reduced to a general second order equation of mixed type and then the uniqueness and existence of strong solution to the generalized Tricomi problem for the system (1) were proved. In this paper we consider the uniqueness of solution for the general case (1). Without loss of generality, assume

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that the following conditions are satisfied

$$\begin{cases} (I) & f=g=0 \\ (II) & A,B,C,\alpha,\beta,\gamma,\delta\in C^1 \\ (III) & \text{for the system (1), there exist solution } u,v\in C^1 \end{cases} \tag{2}$$

We construct an auxiliary function by line integrals

$$\Phi(x,y)=\int_{(x_0,y_0)}^{(x,y)} 2Auve^{\int_{(x_0,y_0)}^{(x,y)} 2\delta(s,t)ds+(\gamma(s,t)-\beta(s,t))dt}dx + ((AC-B^2)u^2+2Buv-v^2)e^{\int_{(x_0,y_0)}^{(x,y)} 2\delta ds+(\gamma-\beta)dt}dy \tag{3}$$

where (x_0,y_0) is an arbitrary point in $\overline{D}$. In order to make the two line integrals in (3) independent of the path of integration, the coefficients in (1) must satisfy the following two conditions:

$$\begin{cases} \text{(i)} \quad 2\frac{\partial}{\partial y}\delta(x,y)=\frac{\partial}{\partial x}(\gamma(x,y)-\beta(x,y)) \\ \text{(ii)} \quad \frac{\partial}{\partial y}[2Auve^{\int_{(x_0,y_0)}^{(x,y)} 2\delta ds+(\gamma-\beta)dt}] \\ \qquad =\frac{\partial}{\partial x}[((AC-B^2)u^2+2Buv-v^2)e^{\int_{(x_0,y_0)}^{(x,y)} 2\delta ds+(\gamma-\beta)dt}] \end{cases}$$

By use of the system (1), the above conditions may be reduced to the following three conditions:

$$\begin{cases} \text{(i)} & 2\frac{\partial}{\partial y}\delta(x,y)=\frac{\partial}{\partial x}(\gamma(x,y)-\beta(x,y)) \\ \text{(ii)} & 2\delta(AC-B^2)=2A\alpha+2B\gamma+(B^2-AC)_x \\ \text{(iii)} & 2\delta B=2(A-1)\gamma+2A_y-2B_x \end{cases} \tag{4}$$

Example 1. For the simple case: $A=1$, $B=0$, $C=k(x,y)$, the conditions in (4) may be written as

$$\begin{cases} \text{(i)} & 2\frac{\partial}{\partial y}\delta(x,y)=\frac{\partial}{\partial x}(\gamma(x,y)-\beta(x,y)) \\ \text{(ii)} & \alpha(x,y)=k(x,y)\delta(x,y)+\frac{1}{2}k_x(x,y) \end{cases} \tag{5}$$

The method of auxiliary function for the equation of mixed type was established by C.S. Morawetz in 1956[2], she considered the Chaplygin's equation. Afterwards we established an auxiliary funtion in general form for the second order equation of mixed type:

$$Aw_{yy}-2Bw_{xy}+Cw_{xx}+Dw_y+Ew_x=0$$

in 1983[3]. The idea constructing this auxiliary function is as follows: multiply the equation by $aw+bw_x+cw_y$ and then reduce it to a divergent form:

$$-[\cdots]_y+\{\cdots\}_x+(\cdots)$$