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THE POINT SPECTRUM OF THE LINEARIZED BOLTZMANN OPERATOR WITH THE EXTERNAL-FORCE TERM IN A BOUNDED DOMAIN

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Abstract We will investigate the point spectrum on the imaginary axis of the linearized Boltzmann operator with an external-force potential in a bounded domain whose boundary is sufficiently smooth. The boundary condition considered is the perfectly reflective boundary condition. The point spectrum on the imaginary axis is only equal to $\{0\}$. However, the null space varies with the common axes of symmetry of the domain and the external-force potential.

Key Words Point spectrum; linearized Boltzmann operator; external-force potential; bounded domain.

Classification 76P05, 45K05.

1. Introduction

The nonlinear Boltzmann equation with an external-force potential $\phi = \phi(x)$ has the form,

$$\partial f/\partial t + \Lambda f = Q(f, f) \tag{1.1}$$

This equation describes the time evolution of rarefied gas acted upon by the external force $\mathbb{F} = -\nabla\phi$. $f = f(t, x, \xi)$ is the unknown function denoting the density of gas particles at time $t \geq 0$, at a point $x \in \Omega$, and with a velocity $\xi \in \mathbf{R}^3$. Ω is a domain $\subseteq \mathbf{R}^3$ in which the rarefied gas is confined. Λ and $Q(\cdot,\cdot)$ are the following operators (see [1–2]):

$$\Lambda \equiv \xi \cdot \nabla_x - \nabla_x\phi \cdot \nabla_\xi$$

$$Q(g, h) \equiv (1/2)\int_{\xi' \in \mathbf{R}^3, s \in S^2} B(\theta, |\xi - \xi'|) \times \{g(\eta)h(\eta') + g(\eta')h(\eta) - g(\xi)h(\xi') - g(\xi')h(\xi)\}d\xi' ds$$

where $g(\eta) = g(t,x,\eta)$, etc., $\eta = \xi - ((\xi - \xi')\cdot s)s$, $\eta' = \xi' + ((\xi - \xi')\cdot s)s$, and $\cos\theta = (\xi-\xi')\cdot s/|\xi-\xi'|$, $s \in S^2$. S^2 denotes the unit sphere whose center is the origin. $B(\theta, V)$ is a nonnegative known function of $(\theta, V) \in [0,\pi]\times[0,+\infty)$. We will impose the following (see [1–2]):

Assumption 1.1 $B(\theta,V)/|\sin\theta\cos\theta| \le c_{1.1}(V + V^{\varepsilon-1})$, where $c_{1.1} > 0$ and $0 < \varepsilon < 1$ are constants independent of (θ, V).

Under this assumption we linearize (1.1) around the absolute Maxwellian state $M \equiv \exp(-E(x,\xi))$, where $E(x,\xi) \equiv \phi(x) + |\xi|^2/2$. Substituting $f = M + M^{1/2}u$ in (1.1), and dropping the nonlinear term, we obtain the linearized Boltzmann equation,

$$\partial u/\partial t = Bu \tag{1.2}$$

where $B \equiv A + e^{-\phi(x)}K$, and $A \equiv -\Lambda + (\exp(-\phi(x)))(-\nu)$. The operator B is the linearized Boltzmann operator. $\nu = \nu(\xi)$ is a multiplication operator, and K is an integration operator with a symmetric kernel. ν and K act on ξ only. These operators satisfy the following (see [1–2]):

Lemma 1.2 (i) *There exists a positive constant $c_{1.2}$ such that for any $\xi \in \mathbf{R}^3$, $0 < \nu(\xi) \le c_{1.2}(1 + |\xi|)$.*

(ii) *K is a self-adjoint compact operator on $L^2(R^3_\xi)$.*

(iii) *$(-\nu + K)$ is a self-adjoint nonpositive operator on $L^2(\mathbf{R}^3_\xi)$.*

(iv) *The point spectrum of $(-\nu + K)$ contains 0, and the null space is spanned by $\xi_j\exp(-|\xi|^2/4)$, $j = 1,2,3$, $\exp(-|\xi|^2/4)$, and $|\xi|^2\exp(-|\xi|^2/4)$, where ξ_j is the j-th component of ξ, $j = 1,2,3$, i.e., $\xi = (\xi_1,\xi_2,\xi_3)$.*

It is important to investigate decaying of solutions of (1.2) (see [3, p.768], [4, p.241], and [5, p.1827]). For this purpose we need to first inspect the point spectrum of B on the imaginary axis and the corresponding eigenspaces. Because we can obtain estimates for the decaying of solutions of (1.2) only in function spaces perpendicular to the eigenspaces corresponding to eigenvalues of B on the imaginary axis (cf. [1–2]).

In [6] we have already investigated this subject when $\Omega = \mathbf{R}^3$, and by making use of the result in [6], we have obtained decay estimates for solutions of (1.2) (cf. [3–4]). In the present paper, we will study that subject when Ω is a bounded domain. The main result is Theorem 4.1. The boundary condition considered is the perfectly reflective boundary condition. We assume that the boundary $\partial\Omega$ is sufficiently smooth, and that the traces upon $\partial\Omega$ of functions contained in the domain of B are square-integrable with respect to some measure on $\partial\Omega \times \mathbf{R}^3$.

In [6], the eigenvalues of B and the corresponding eigenfunctions have only to satisfy the following:

$$\mu v = Bv \tag{1.3}$$

In this paper, we obtain μ and v which satisfy (1.3), and moreover we need to examine whether v satisfies the perfectly reflective boundary condition or not. The forms of eigenfunctions of B are heavily restricted by this fact, and hence we have to perform more complicated calculations than those in [6].