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A NEW COMPLETELY INTEGRABLE LIOUVILLE'S SYSTEM PRODUCED BY THE MA EIGENVALUE PROBLEM

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Abstract Under the constraint between the potentials and eigenfunctions, the Ma eigenvalue problem is nonlinearized as a new completely integrable Hamiltonian system $(\mathbf{R}^{2N}, dp\wedge dq, H)$:

$$H = \frac{1}{2}\alpha \langle \Lambda q, p\rangle - \frac{1}{2}\alpha_3 \langle q, q\rangle + \frac{\alpha}{4\alpha_3\eta} \langle q, p\rangle \langle p, p\rangle$$

The involutive solution of the high-order Ma equation is also presented. The new completely integrable Hamiltonian systems are obtained for DLW and Levi eigenvalue problems by reducing the remarkable Ma eigenvalue problem.

Key Words Ma eigenvalue problem; Bargmann constraint; involutive system; involutive solution.

Classification 35Q20.

1. Introduction

An elegant geometric theory of finite-dimensional Hamiltonian systems has been built since the late sixties[1–3]. The Liouville-Arnold theorem is beautiful. According to this theorem a Hamiltonian system with N freedoms is completely integrable if it possesses N involutive functional systems. The involutive functional systems are obtained in general by the spectral technique associated with nonlinear evolution equations, and thus new completely integrable systems are produced[4,5]. Flaschka[6] pointed out an important principle to produce finite-dimensional integrable systems by constraining the infinite-dimensional integrable systems on a finite-dimensional invariant manifold. Recently, Cao Cewen[7,8] has presented a systematic approach to get finite-dimensional integrable systems by the nonlinearization of Lax pair of soliton equations and has successfully found many finite-dimensional completely integrable systems.

This article is divided into four sections. In the next section, we present a new completely integrable Hamiltonian system in Liouville's sense which is generated through nonlinearization of the Ma eigenvalue problem. In Sec. 3 the involutive solution of the high-order Ma equation is obtained. In Sec. 4 we regard the Dispersive Long Wave (DLW) and the Levi eigenvalue problems as reductions of the remarkable Ma eigenvalue problem. Based on this fact, we obtain two new completely integrable Hamiltonian systems in Liouville's sense which are produced by the DLW and Levi eigenvalue problems respectively.

2. Nonlinearization of the Ma Eigenvalue Problem and a Finite-Dimensional Involutive System

Consider the Ma eigenvalue problem[9]

$$\varphi_x = U\varphi = \begin{bmatrix} \alpha_1\lambda + \alpha_4 s & r \\ \alpha_3 & \alpha_2\lambda + s \end{bmatrix}\varphi, \quad (\alpha_1 - \alpha_2)\alpha_3(\alpha_4 - 1) \neq 0 \tag{2.1}$$

where α_i $(i = 1, 2, 3, 4)$ are constants, λ is an eigenparameter, $\varphi = (\varphi_1, \varphi_2)^T$, $\varphi_x = \partial\varphi/\partial x$, $r(x,t)$ and $s(x,t)$ are potential functions, $x \in \Omega$. The underlying interval Ω is $(-\infty, +\infty)$ or $(0, T)$ under the decaying conditions at infinitely or periodic condition respectively.

By making the transformation $\varphi = y\exp\left[\frac{1}{2}(\alpha_1+\alpha_2)\lambda x + \frac{1}{2}(\alpha_4+1)\partial^{-1}s\right]$ and its inverse $y = \varphi\exp\left[-\frac{1}{2}(\alpha_1+\alpha_2)\lambda x - \frac{1}{2}(\alpha_4+1)\partial^{-1}s\right]$, (2.1) is equivalent to the eigenvalue problem

$$y_x = My = \begin{bmatrix} \frac{1}{2}[\alpha\lambda + (\alpha_4-1)s] & r \\ \alpha_3 & -\frac{1}{2}[\alpha\lambda + (\alpha_4-1)s] \end{bmatrix} y \tag{2.2}$$

where $\alpha = \alpha_1 - \alpha_2 \neq 0$, $\partial^{-1}\partial = \partial\partial^{-1} = 1$.

Proposition 2.1 *Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be N different eigenvalues of Eq.(2.2), $(q_j, p_j)^T$, $(j = 1, 2, \cdots, N)$ be the corresponding eigenfunctions. Then the functional gradient $\nabla\lambda_j$ of λ_j is*

$$\nabla\lambda_j = \begin{bmatrix} \delta\lambda_j/\delta r \\ \delta\lambda_j/\delta s \end{bmatrix} = \begin{bmatrix} -\alpha^{-1}p_j^2 \\ -\alpha^{-1}(\alpha_4 - 1)q_jp_j \end{bmatrix} \cdot \left(\int_\Omega q_jp_j dx\right)^{-1} \tag{2.3}$$

Proof See Sec II, Ref. 10.