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INVERSE SCATTERING FOR THE PROBLEM WITH IMPEDANCE-TYPE BOUNDARY

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Abstract This paper deals with the inverse scattering problems for the Helmholtz equation with impedance boundary condition. It aims at reconstructing the unknown impedance coefficient from the knowledge of scattered wave fields. We generalize the concept of classic solution (CS) to optimal solution (OS) by a nonlinear optimization problem. Then, based on potential theory, we establish an inversion procedure to get the approximation of OS which is defined as the regularized solution (RS) in this paper. The convergence result for RS is proven from which one can get OS and CS stably and efficiently.

Key Words Inverse problem; optimal solution; convergence.
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1. Introduction

The inverse scattering problem based on the Helmholtz equation is of great importance both to physics application and to mathematical theory. Generally speaking, such kind of problems requires to recover the unknown ingredients such as scatter shape, equation coefficient and so on from the knowledge of scattered wave fields. Due to the ill-posedness of these problems, people often reformulate such inverse problems as some optimization problems under some constraints, then solve them by optimal control techniques. However, most of the methods for approximately getting the optimal solution are of an interactive nature and require the solution of corresponding direct problem at each iteration step. The disadvantage of such methods includes a great number of computation and the nonconvergence of approximate solution sequence.

In this paper, we develop an inversion procedure to get the optimal control solution for recovery of the boundary impedance coefficient. By applying the potential theory, we firstly transform the inverse problem into a nonlinear operator equations, and avoid solving the direct problem in each iteration step. Therefore, the number of calculation is weaken greatly. Then, we reformulat the nonlinear operator equation (it may not have classic solution) as a nonlinear optimizatiom problem. For this optimal problem, we prove the existence and convergence of regularized solution. The relation between the regularized solution for nonlinear equations and the classic solution for original inverse problem (if it exists) is also discussed. Our work provides a stable and convergent approach to reconstruction of boundary impedance from general $u^\infty(\hat{x})$ with error from the far-field pattern.

Now, let us consider the mathematical model. Suppose the incident plane wave $u^i(x) = e^{ikd\cdot x}$, where unit vector $d \in R^2$ represents the incident direction, is scattered by an obstacle $D \subset R^2$ with impedance-type boundary ∂D. Denote by $u^s(x)$ the corresponding scattering wave field. Then the total wave field

$$u(x) = u^i(x) + u^s(x) \tag{1.1}$$

is govern by

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } R^2\backslash\bar{D} \\ \dfrac{\partial u}{\partial \gamma} + ik\lambda(x)u = 0, & \text{on } \partial D \end{cases} \tag{1.2}$$

where $\lambda(x)$ is called the boundary impedance coefficient. Furthermore, $u^s(x)$ should satisfy the Sommerfeld condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x| \tag{1.3}$$

uniformly in all directions $\dfrac{x}{|x|}$. Since $u^i(x) = e^{ikd\cdot x}$ is an entire solution of the Helmholtz equation in R^2, $u^s(x)$ can be determined by

$$\begin{cases} \Delta u^s + k^2 u^s = 0, & \text{in } R^2\backslash\bar{D} \\ \dfrac{\partial u^s}{\partial \gamma} + ik\lambda(x)u^s = f(x), & \text{on } \partial D \\ \lim\limits_{r\to\infty} \sqrt{r}\left(\dfrac{\partial u^s}{\partial r} - iku^s\right) = 0, & r = |x| \end{cases} \tag{1.4}$$

where $f(x) = -\left\{\dfrac{\partial u^i}{\partial \gamma} + ik\lambda(x)u^i\right\}$.

The physics background for scattering problem (1.4) can be found in [1]. With the aid of Green expression for $u^s(x)$ and the singularity analysis for fundamental solution $\Phi(x,y)$ to 2-D Helmholtz equation, it is well-known that the scattered wave has the asymptotic behavior

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}}\left[u^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right], \quad \hat{x} = \frac{x}{|x|} \in \Omega \text{ (unit cycle)} \tag{1.5}$$

which holds uniformly for $|x| \to \infty$. $u^\infty(\hat{x})$ is called the far-field pattern for scattering field $u^s(x)$. Furthermore, there exists a one-to-one correspondance between $u^s(x)$ and $u^\infty(\hat{x})$ in terms of Rellich lemma ([2]). Therefore, the inverse problem can be stated as