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GRADIENT CATASTROPHE IN THE CLASSICAL SOLUTIONS OF NONLINEAR HYPERBOLIC SYSTEMS

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Abstract Classical solutions of hyperbolic systems, generally, collapse in finite time, even for small and smooth initial data. Here, we consider a type of these systems and prove a blow up result.
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1. Introduction

In this work, we are concerned with the Cauchy problem for one-dimensional first order quasilinear hyperbolic systems. In fact this problem has been discussed by many authors and several results concerning existence and formation of singularities have been established. Here, we consider a strictly hyperbolic system of the form:

$$\begin{cases} u_t(x,t) = a(u(x,t), v(x,t))v_x(x,t) \\ v_t(x,t) = b(u(x,t), v(x,t))u_x(x,t) \end{cases} \tag{1.1}$$

where a subscript denotes a partial derivative to the relevant variable; $x \in \mathbb{R}$, and $t > 0$.

In is indeed well known that, generally, classical solutions for such systems break down in finite time, even for smooth and small initial data. Lax [1] and MacCamy and Mizel [2] studied the system, for a depending on v only and $b \equiv 1$, and showed that the solutions blow up in a finite time, even if the initial data are smooth and small. Note in this particular case, the system is reduced to the nonlinear wave equation. For a and b depending on v only (or u only), similar results were established, for systems with dissipation, by Slemrod [3], Kosinskii [4] and Messaoudi [5].

It is interesting to mention that global existence for the system considered in [6] has been established by Nishida [6]. Also, Aregba and Hanouzet [7] and Tartar [8] have considered a class of semilinear hyperbolic system and proved some global existence and blow-up results.

In this paper, we study the system (1.1) together with initial data and show that a result similar to the one in [1], [2] can be obtained. The proof will be based on the use of characteristics and the theory of linear first-order partial differential equations.

2. Local Existence

We consider the following problem

$$u_t(x,t) = a(u(x,t), v(x,t))v_x(x,t) \tag{2.1}$$

$$v_t(x,t) = b(u(x,t), v(x,t))u_x(x,t), \quad x \in \mathbb{R}, t \geq 0 \tag{2.2}$$

$$u(x,0) = u_0(x), v(x,0) = v_0(x), \quad x \in \mathbb{R} \tag{2.3}$$

Proposition 2.1 *Assume that a and b are C^2 strictly positive functions and let u_0 and v_0 in $H^2(\mathbb{R})$ be given. Then the problem (2.1)–(2.3) has a unique local solution (u, v), on a maximal time interval $[0, T)$, satisfying*

$$u, v \in C([0,T), H^2(\mathbb{R})) \cap C^1([0,T), H^1(\mathbb{R})) \tag{2.4}$$

This result can be proved by either using a classical energy argument [9] or the nonlinear semigroup theory [10].

Remark 2.1 A similar result can also be established for a and b strictly negative.

Remark 2.2 u, v are in $C^1(\mathbb{R} \times [0, T))$ by the Sobolev embedding theorem.

Remark 2.3 A local existence result is also available for higher-dimensional hyperbolic systems (See [9, 10]).

Remark 2.4 If a and b are smooth enough and u_0 and v_0 are in $H^k(\mathbb{R})$, then the solution

$$u, v \in \bigcap_{i=1}^{k} C^i([0,T), H^{k-i}(\mathbb{R})) \tag{2.5}$$

3. Formation of Singularities

In this section, we state and prove our main result. We first begin with a lemma that gives uniform bounds on the solution.

Lemma 3.1 *Assume that a and b are as in the proposition 2.1. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any u_0, v_0 in $H^2(\mathbb{R})$ satisfying*

$$|u_0(x)| < \delta, \quad |v_0(x)| < \delta, \quad x \in \mathbb{R} \tag{3.1}$$

the solution satisfies

$$|u(x,t)| < \varepsilon, \quad |v(x,t)| < \varepsilon, \quad x \in \mathbb{R}, \quad t \in [0,T) \tag{3.2}$$