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NEUMANN PROBLEM OF QUASILINEAR ELLIPTIC EQUATIONS WITH LIMIT NONLINEARITY IN BOUNDARY CONDITION

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Abstract This paper is concerned with the existence of solutions to the equation $D_j(a^{ij}(x,u)D_iu) - \frac{1}{2}D_s a^{ij}(x,u)D_iuD_ju + \lambda u = 0$ on a bounded domain under the Neumann boundary condition $a^{ij}(x,u)D_iu\gamma_j = |u|^{\frac{2}{n-2}}u$.
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1. Introduction

Let Ω be a smooth bounded domain in $R^n (n \geq 3)$, we are concerned with the existence of solutions to the quasilinear elliptic problem

$$D_j(a^{ij}(x,u)D_iu) - \frac{1}{2}D_s a^{ij}(x,u)D_iuD_ju + \lambda u = 0 \quad \text{in } \Omega \tag{1.1}$$

$$a^{ij}(x,u)D_iu\gamma_j = |u|^{2^+-2}u \quad \text{on } \partial\Omega \tag{1.2}$$

where $2^+ = \dfrac{2n-2}{n-2}$, $\gamma = \gamma(x) = (\gamma_1, \gamma_2, \cdots \gamma_n)$ is the unit outward normal to $\partial\Omega$ at x. Assume that each $a^{ij} : \Omega \times R \to R$ is measurable in x for all $s \in R$ and of C^1 in s for a.e. $x \in \Omega$ with $a^{ij} = a^{ji}$. Supose also that a^{ij} satisfies:

(a.1) there exists a constant $C > 0$ such that

$$|a^{ij}(x,s)| \leq C, |D_s a^{ij}(x,s)| \leq C, \text{ a.e. } x \in \Omega, \forall s \in R$$

(a.2) $a^{ij} = a^{ji}$, and there exists a constant $\nu > 0$ such that

$$a^{ij}(x,s)\xi_i\xi_j \geq \nu|\xi|^2, \text{ a.e. } x \in \Omega, \forall s \in R, \xi = (\xi_i) \in R^n$$

(a.3) $sD_s a^{ij}(x,s)\xi_i\xi_j \geq 0$, a.e. $x \in \Omega, \forall s \in R, \xi = (\xi_i) \in R^n$;

(a.4) there is a constant $a_\infty > 0$ such that

$$\lim_{|s|\to\infty} a^{ij}(x,s) = a_\infty\delta^{ij}, \quad \lim_{|s|\to\infty} sD_s a^{ij}(x,s) = 0, \text{ uniformly in a.e. } x \in \Omega$$

We say u is a weak solution of (1.1), (1.2) if $u \in H^1(\Omega)$ and

$$\int_\Omega a^{ij}(x,u)D_iuD_jvdx + \frac{1}{2}\int_\Omega D_sa^{ij}(x,u)D_iuD_juvdx - \int_\Omega \lambda uvdx - \int_{\partial\Omega} |u|^{2^+-2}uvd\sigma = 0, \quad \forall v \in C^\infty(\Omega) \tag{1.3}$$

(1.3) has a variational form, the corresponding functional is

$$J(u) = \frac{1}{2}\int_\Omega a^{ij}(x,u)D_iuD_judx - \frac{\lambda}{2}\int_\Omega u^2dx - \frac{1}{2^+}\int_{\partial\Omega} |u|^{2^+}d\sigma, \quad u \in H^1(\Omega)$$

and $J : H^1(\Omega) \to R$ is continuous, but not differentiable. Let

$$J'(u)[v] = \lim_{t\to 0}\frac{J(u+tv) - J(u)}{t}$$

then for every $u \in H^1(\Omega)$ and $v \in C^\infty(\Omega)$ the limit exists and

$$J'(u)[v] = \int_\Omega a^{ij}(x,u)D_iuD_jvdx + \frac{1}{2}\int_\Omega D_sa^{ij}(x,u)D_iuD_juvdx - \int_\Omega \lambda uvdx - \int_{\partial\Omega} |u|^{2^+-2}uvd\sigma$$

Hence $u \in H^1(\Omega)$ satisfies $J'(u)[v] = 0$ for each $v \in C^\infty(\Omega)$ if and only if u is the weak solution of (1.1) (1.2).

In order to look for such weak solutions, one may utilize Critical Point Theory for Continuous Functionals, developed by G. Katriel and A. Ioffe, M. Degivanni and M. Marzocchi et al., see [1–3]. In our paper, however, instead of the Critical Point Theory for Continuous Functionals, we use the Ekeland Variational Principle to search for the concrete P.S. sequences ((CPS) sequences) for the continuous functional J, and then discuss the convergence of these (CPS) sequences. As we all know, the embedding $u \in H^1(\Omega) \hookrightarrow L^{2^+}(\partial\Omega)$ is not compact, which causes new difficulties in treating the problem (1.1), (1.2). Similar to the semilinear case, we can prove that any (CPS) sequence of J is convergent under a certain level, and based on this fact, we obtain the existence of solutions to the problem (1.1) (1.2).

The Neumann problem of semilinear elliptic equations with critical growth was studied by Wang [4], in which some delicate computations was made to estimate the critical value of the associated functional. A typical fact is that the best constant S in the Sobolev inequality

$$\|u\|_{2^*,R^n} \le S\|Du\|_{2,R^n}$$

is achieved by a family of the functions $u_\varepsilon(x) = \varepsilon^{(n-2)/4}(\varepsilon + |x|^2)^{-(n-2)/2}$. For Neumann problem of semilinear elliptic equations with limit nonlinearity in boundary condition, the authors respectively made study in [5,6]. Both arguments in [5] and [6]