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VERY WEAK SOLUTIONS OF p- LAPLACIAN TYPE EQUATIONS WITH VMO COEFFICIENTS*

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Abstract In this note we obtain a new *a priori* estimate for the very weak solutions of p-Laplacian type equations with VMO coefficients when p is close to 2, and then prove that the very weak solutions of such equations are the usual weak solutions. Our approach is based on the Hodge decomposition and the L^p-estimate for the corresponding linear equations. And this also provides a simpler proof for the results in [1].

Key Words p-Laplacian type; VMO coefficients; very weak solution.
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1. Introduction

Let $1 < p < \infty$. We consider the very weak solutions of the quasilinear equation

$$\operatorname{div}((ADu \cdot Du)^{(p-2)/2} ADu) = 0 \text{ in } \mathbf{R}^n \tag{1.1}$$

where $A = (A_{ij}(x))_{n\times n}$ is a symmetric matrix with measurable coefficients satisfying the uniform ellipticity condition

$$\nu^{-1}|\xi|^2 \le A(x)\xi \cdot \xi \le \nu|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbf{R}^n$ and almost every $x \in \mathbf{R}^n$. Here $\nu \ge 1$ is a constant and the dot denotes the standard inner product of $\mathbf{R}^n$.

The equation (1.1) arises naturally in many different contexts. In the case $p = n$ the equation (1.1) plays a key role in the theory of quasiconformal mappings. If A is the identity matrix, then it reduces to the known p-harmonic equation. If $p = 2$, it is a linear equation.

We recall the definition of the very weak solution for the equation (1.1).

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Definition 1.1 *A function $u \in W^{1,r}_{\text{loc}}(\mathbf{R}^n), \max\{1, p-1\} \le r \le p$, is called a very weak solution of the equation* (1.1) *if*

$$\int_{\mathbf{R}^n} (ADu \cdot Du)^{(p-2)/2} ADu \cdot D\psi dx = 0 \tag{1.3}$$

for every $\psi \in C_0^\infty(\mathbf{R}^n)$.

We are concerned with the question whether the very weak solutions of the equation (1.1) are the usual weak solutions, that is, whether the very weak solutions belong to the Sobolev space $W^{1,p}_{\text{loc}}$. Generally speaking, the answer is not true even in the linear case $p = 2$ (see [2]). However, there is still some hope when the coefficients in the equation (1.1) satisfy certain nice condition as well as the uniformly ellipticity condition (1.2). The case with discontinuous coefficients is much interesting. Therefore, we are keen on considering the problem with the discontinuous coefficients. A natural weakness of the case with smooth coefficients is to assume that the coefficients of the matrix A are of vanishing mean oscillation (VMO).

We say that a locally integrable function f is of bounded mean oscillation (BMO) if $⨍_B|f - f_B|dy$ is uniformly bounded as B ranges over all balls in $\mathbf{R}^n$, here $f_B = ⨍_B f(y)dy = |B|^{-1}\int_B f(y)dy$ denotes the integral mean over the ball B. If, in addition, we require that these averages tend to zero uniformly as the radii tend to zero, we say that f is of vanishing mean oscillation and denotes $f \in$ VMO, see [3]. Uniformly continuous functions and $W^{1,n}$ functions are of vanishing mean oscillation. In general, the functions of vanishing mean oscillation need not be continuous.

We state our main result as below.

Theorem 1.2 *Suppose that $A_{ij} \in VMO$, $i, j = 1, \cdots, n$, satisfy the condition* (1.2). *For every $\eta, 0 < \eta < 1$, there is a positive number $\delta = \delta(\eta, n, \nu, A_\#) \le \eta$ such that every very weak solution $u \in W^{1,r}_{\text{loc}}, 1 + \eta \le r \le p$, of the equation* (1.1) *is the usual weak solution provided that $|p - 2| \le \delta$. Namely, $u \in W^{1,p}_{\text{loc}}$ and it satisfies the equality* (1.3). *Here $A_\#$ denotes the VMO modulus of the coefficients $A_{ij}, i, j = 1, \cdots, n$.*

When $p = 2$, we have

Corollary 1.3 *Suppose that $A_{ij} \in VMO$, $i, j = 1, \cdots, n$, satisfy the condition* (1.2). *Then every very weak solution $u \in W^{1,r}_{\text{loc}}$ with $r > 1$ of equation*

$$\operatorname{div}(ADu) = 0$$

is the usual weak solution. Namely, $u \in W^{1,2}_{loc}$ and it satisfies

$$\int_{\mathbf{R}^n} ADu \cdot D\psi dx = 0$$

for every $\psi \in C_0^\infty(\mathbf{R}^n)$.

However, the above result is not true even in the linear case if the coefficents $A_{ij}(x), i, j = 1, \cdots, n$ are only bounded and measurable since Serrin provided a well-known counter example more than thirty years ago ([2]). In the beginning of nineties,