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GINZBURG-LANDAU VORTICES IN INHOMOGENEOUS SUPERCONDUCTORS*

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Abstract We study the vortex convergence for an inhomogeneous Ginzburg-Landau equation, $-\Delta u = \varepsilon^{-2} u(a(x) - |u|^2)$, and prove that the vortices are attracted to the minimum point b of $a(x)$ as $\varepsilon \to 0$. Moreover, we show that there exists a subsequence $\varepsilon \to 0$ such that u_ε converges to u strongly in $H^1_{loc}(\bar{\Omega} \setminus \{b\})$.

Key Words Vortex; Ginzburg-Landau equation; elliptic estimate; H^1-strong convergence.

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1. Introduction

Let $\Omega \subset R^2$ be a smooth, bounded and simply connected domain occupied by an inhomogeneous type II-superconducting material. Due to the inhomogeneities, the equilibrium density of superconducting electrons is not a constant, but a positive smooth function on Ω. Denote it by $a = a(x)$. In the steady state, this model, proposed by Likharev [1], is characterized by the following equations:

$$\begin{cases} \Delta u = -\dfrac{u}{\varepsilon^2}(a(x) - |u|^2), & \text{in } \Omega, \\ u(x) = g_1(x), & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Also see [2] and [3, 4] for more-complicated time-dependent model. In the equation (1.1), we suppose that $g_1 : \partial\Omega \longrightarrow R^2$ is smooth and satisfies

$$|g_1(x)| = \sqrt{a(x)} \quad on \quad \partial\Omega. \tag{1.2}$$

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It is easy to see that there exists at least a smooth solution u_ε to (1.1) for each $\varepsilon > 0$. In fact, the minimizer of the functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}(a(x) - |u|^2)^2\right) dx \tag{1.3}$$

on $H^1_{g_1}(\Omega) = \{u \in H^1(\Omega, R^2) : u = g_1, \quad \text{on } \partial\Omega\}$ is a solution of (1.1). Such a u is called a minimum solution to the equation (1.1).

We are interested in the asymptotic behavior of u_ε as $\varepsilon \longrightarrow 0$. In the case where $a(x) = 1$ for any $x \in \Omega$, this problem was studied by Bethuel, Brezis and Hélein in [5], Struwe in [6] and Lin in [7]. In this case, the value $d_1 = deg(g_1, \partial\Omega)$, the Brouwer degree of g_1 considered as a map from $\partial\Omega$ into S^1, plays a crucial role. When $d_1 = 0$, the results in [8] show that the minimum solution converges to a smooth harmonic map from Ω into S^1 which equals to g_1 on $\partial\Omega$; when $d_1 \neq 0$, the situation is much more delicate, and singularities and vortices appear. See [5-7] for the details. In the case where $a(x)$ is not a constant, for example, $a(x)$ has a strict minimum in Ω, Chapman and Richardson in a recent paper [2] used a matched asymptotic method to derive formally that the vortices, i.e., the points at which the solution for the equation (1.1) (more generally, for a time dependent equation whose stead-state is (1.1)) equals to zero, are attracted to the the minimum of $a(x)$. In [9], the authors tried to prove this phenomenon rigorously. But no H^1-strong convergence has been obtained.

In this paper, we will prove a H^1-strong convergence result. To state this main result, we set

$$g(x) = \frac{g_1(x)}{\sqrt{a(x)}}, \qquad d = \deg(g, \partial\Omega).$$

Then, under the hypothesis (h_2) below, one has

$$d = \frac{1}{2\pi}\int_{\partial\Omega} \frac{g}{|g|^2} \wedge \frac{\partial g}{\partial T} = \frac{1}{2\pi}\int_{\partial\Omega} \frac{g_1}{|g|_1^2} \wedge \frac{\partial g_1}{\partial T} = d_1.$$

We may assume $d > 0$ since the case $d < 0$ is completely similar to the case $d > 0$ and no vortex is expected to be appeared in the case $d = 0$. For simplicity, we only consider the case where $a(x)$ has a unique minimum; more precisely, assume that $b = (b_1, b_2) \in \Omega$ is the only minimum point of $a(x)$ in $\bar{\Omega}$. Moreover, we will suppose that

$(\mathbf{h_1})$ Ω is starshaped with respect to the point b;

$(\mathbf{h_2})$ $a \in C^3(\bar{\Omega})$ and $a(x) > 0$ for all $x \in \bar{\Omega}$;

$(\mathbf{h_3})$ $\nabla a(x) \cdot (x - b) > 0$ for all $x \neq b$, $x \in \bar{\Omega}$ and the matrix function $M = (m_{kj})$, where $m_{kj} = \frac{\partial a(x)}{\partial x_k}(x_j - b_j)$, $k, j = 1, 2$, is semi-positive definite for all $x \in \Omega$;

or

$(\mathbf{h_3'})$ $(\nabla a(x)) \cdot (x - b) - 2|(\nabla a(x))^\perp \cdot (x - b)| > 0$ for any $x \in \bar{\Omega}$ and $x \neq b$, where

$$(\nabla a(x))^\perp = \left(-\frac{\partial a(x)}{\partial x_2}, \frac{\partial a(x)}{\partial x_1}\right).$$