J. Partial Diff. Eqs. **16**(2003), 275–288
©International Academic Publishers Vol.16, No.3

THE CAUCHY PROBLEM FOR THE GENERALIZED KORTEWEG-DE VRIES-BURGERS EQUATION IN $\dot{H}^{-s}$

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(Received Nov. 11, 2002)

Abstract The Cauchy problem for the generalized Korteweg-de Vries-Burgers equation is considered and the local existence and uniqueness of solutions in $L^q(0,T;L^p) \cap L^\infty(0,T;\dot{H}^{-s})(0 \le s < 1)$ are obtained for initial data in $\dot{H}^{-s}$. Moreover, the local solutions are global if the initial data are sufficiently small in critical case. Particularly, for $s = 0$, the generalized Korteweg-de Vries-Burgers equation satisfies the energy equality, so the initial data can be arbitrarily large to obtain the global solution.
Key Words generalized Korteweg-de Vries-Burgers equation; Cauchy problem; space-time dual estimates; $\dot{H}^{-s}$ solution.
2000 MR Subject Classification 35Q53, 35G25, 35A05.
Chinese Library Classification O175.28, O175.29.

1. Introduction

The subject of this paper is the Cauchy problem for the generalized Korteweg-de Vries-Burgers equation

$$\partial_t u - \nu \triangle u + \sum_{k=1}^{n} \frac{\partial^3 u}{\partial x_k^3} + F(u) = 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

where $x \in I\!R^n, t \geqslant 0$, and ν is a fixed non-negative constant, $\triangle$ denotes the Laplacian on $I\!R^n$. Denote $F(u) = \nabla \cdot \mathbf{f}(u)$, where

$$\mathbf{f}(u) = (f_1, f_2, ..., f_n), f_k \in C^1, f_k(0) = 0, |f_k'(u)| \le C|u|^\alpha, (k = 1, ..., n). \tag{1.3}$$

This equation, used as a classical model on long wave propagation, arises in modeling unidirectional propagation of planar waves in observing interaction of nonlinearity, dispersion and dissipation phenomena.

In the case $n = 1$, there are some recent considerations (see [1, 2]) to the following problem:

$$\partial_t u - \nu\partial_x^2 u + \partial_x^3 u + u\partial_x u = 0, \tag{1.4}$$

$$u(x, 0) = u_0(x). \tag{1.5}$$

The result of existence, uniqueness, continuous dependence and smoothness of solutions for (1.4) and (1.5) was established by Bona and Smith [3] under the hypothesis that $u_0 \in H^s(\mathbb{R})$, where $s \geqslant 2$. A classical perturbation argument allows us to weaken this assumption to $s \geq 0$. In [4] a general case was dealt with. G.Karch [2] showed that (1.4) and (1.5) have a unique mild solution $u \in C([0, \infty); L^2(\mathbb{R}^n))$ if $u_0 \in L^2(\mathbb{R}^n)$. For any $s \geqslant 0$, it belongs to $C_{\text{loc}}((0, \infty); H^s(\mathbb{R}^n))$. Moreover, $u \in C_{\text{loc}}((0, \infty); W^{l,1}(\mathbb{R}^n))$ for each non-negative integer l if $u_0 \in L^1(\mathbb{R}^n)$. G.Karch Also investigated the behavior of the solution to the Cauchy problem (1.4) and (1.5) as t tends to infinity.

In the present paper, we study the solution of (1.1) and (1.2). First, we shall give some space-time dual estimates for the solution of linear generalized Korteweg-de Vries-Burgers equation. Moreover, we establish the local and global existence and uniqueness of the solution of (1.1) and (1.2).

We construct a suitable space $L^q(0, T; L^p)$, for some p and q, in which there is a unique local solution of (1.1) and (1.2). It is necessary to point out that the semigroup of the solution for the generalized Korteweg-de Vries-Burgers equation has two semigroups' properties in which one is the semigroup of the solution for parabolic equation, and the other is the one for Schrödinger equation. Each semigroup has its own peculiar properties. The global existence of solution relies upon a delicate balance between these two semigroups and the growth of the nonlinearity. In the sequel, we will show the local and global existence and uniqueness of solution of the generalized Korteweg-de Vries-Burgers equation for the Cauchy data in $\dot{H}^{-s}(0 \leq s < 1)$.

Notations Let $L^p := L^p(\mathbb{R}^n)(1 \leq p \leq \infty)$ denote the Lebesgue space on $\mathbb{R}^n$, $L^q(L^p) := L^q(0, T; L^p(\mathbb{R}^n))$ is the space-time Lebesgue space with norm $\|\cdot\|_{L^q(L^p)}$. We define

$$A(t) := \mathcal{F}^{-1} e^{it\sum_{k=1}^n \xi_k^3} \mathcal{F}, \quad B(t) := \mathcal{F}^{-1} e^{-\nu t|\xi|^2} \mathcal{F}, \tag{1.6}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse transform. The composition of $A(t)$ and $B(t)$ is defined by

$$M(t) := A(t)B(t) = \mathcal{F}^{-1} e^{it\sum_{k=1}^n \xi_k^3 - \nu t|\xi|^2} \mathcal{F}. \tag{1.7}$$

Moreover,

$$M^*(t) := \mathcal{F}^{-1} e^{-it\sum_{k=1}^n \xi_k^3 - \nu t|\xi|^2} \mathcal{F}. \tag{1.8}$$

Then, we define

$$\mathcal{A}F(u) := \int_0^t M(t - \tau)F(u(x, \tau))d\tau = \int_0^t M(t - \tau)\nabla \cdot \mathbf{f}(u)(x, \tau)d\tau. \tag{1.9}$$