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THE CLASSICAL SOLUTIONS TO A CAUCHY PROBLEM OF PARABOLIC TYPE COUPLED WITH OPERATORS

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Abstract We consider the equation

$$u_t = Tr[B(x, t, Du, \Phi u)D^2u] + F(x, t, u, Du, \Phi u, \Psi u)$$

where Φ and Ψ are vector-valued mappings. We obtain the existence and uniqueness of classical solution to the equation for a ϵ- periodic initial data. The problem is naturally arisen from image denoising.
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1. Introduction

The problem of denoising is fundamental for image reconstruction. Image denoising is a technique that enhances images by removing or diminishing degradations that may be present. Since it is usually impossible to identify the kind of noise involved in a given real image, some assumption has to be made. Let $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ be an intensity image, and I be the observed or degenerate image. In this paper we consider the noisy image model

$$I(x) = u(x) + n(x),\ x \in \Omega,$$

where Ω is the image domain, which usually is a rectangle in two-dimensional case, and n denotes a white additive Gaussian noise, i.e.,$n \sim N(0, \sigma)$. Then our problem is to recover u knowing I.

It can be seen that Total Variation(TV) methods are very effective for recovering "blocky", possibly discontinuous image from noisy data [1]. Rudin, Osher and Fatemi[2] illuminatingly introduced the constrained minimization problem

$$\min \int_\Omega |\nabla u| dx \qquad \text{subject to } \|u - I\|^2 = \sigma^2 \text{ and } \int_\Omega u = \int_\Omega I,$$

which is naturally linked to an unconstrained minimization problem

$$\min E(u) \equiv \int_\Omega |\nabla u| + \frac{\lambda}{2}|u - I|^2$$

for a given Lagrange multiplier λ.

To smooth images more selectively, Strong and Chan[3] proposed spatially adaptive TV scheme

$$\min \int_\Omega \{\alpha(x)|\nabla u| + \int_\Omega \frac{\lambda}{2}|u - I|^2\} dx,$$

where $\alpha(x)$ is chosen to be inversely proportional to the likelihood of the presence of an edge. $\alpha(x)$ is ideally a differentiable function having value zero on the edges and value one in the homogeneous regions. In order to avoid the presence of $|\nabla u|$ as a denominator in the corresponding evolution equation, and then to both improve the numerical implementation and obtain sound mathematical basis, Chen, Vermuri and Wang[4] reconsidered the Strong-Chan model and presented a nonlinear diffusion equation supplemented with reactive term for achieving edge preserving smoothing:

$$u_t = g(\nabla G_\sigma * u)|\nabla u| \text{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \nabla g(\nabla G_\sigma * u) \cdot \nabla u - \lambda|\nabla u|(u - I),$$
$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0, \qquad u(x,0) = I,$$

where $G_\sigma(x) = (1/4\pi\sigma)\exp\left\{-\frac{|x|^2}{4\sigma}\right\}$ is a Gaussian kernel and $g(s) = 1/(1 + K|s|^2)$ is a non-increasing real valued function (for $s \geq 0$),and $K > 0$ is constant.

Noting that the Riemann norm $|\cdot|$ in $E(u)$ is not differentiable at zero, Vogel, Oman [1] considered the minimization of the TV-penalized least square functional

$$\text{minimize } f(u) = \frac{1}{2}\|u - I\|^2 + \alpha J_\beta(u) \tag{1.1}$$

where

$$J_\beta(u) = \int_\Omega \sqrt{|\nabla u|^2 + \beta^2},$$

in which α and β are (typically small) positive parameters and Ω is the domain of the image.

We shall then, in our paper, propose the following TV-penalized least square minimization problem

$$\min_u E_\beta(u) =: \frac{\lambda}{2}\int_\Omega |u - I|^2 dx + \int_\Omega \alpha(x)\sqrt{|\nabla u|^2 + \beta^2} dx. \tag{1.2}$$