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# BLOWUP FOR SYSTEMS OF SEMILINEAR WAVE EQUATIONS WITH SMALL INITIAL DATA*

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**Abstract** This paper deals with the Cauchy problem for the system of semilinear wave equations with small initial data. We give the upper bounds for the lifespan of the classical solution to the systems.
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## 1. Introduction and Main Result

In this paper, we consider the system of semilinear wave equations with small initial data

$$\begin{cases} u_{tt}(x,t) - a^2\triangle u(x,t) = |v_t(x,t)|^p & \text{in } \mathbb{R}^n \times \{t \geq 0\}, \\ v_{tt}(x,t) - \triangle v(x,t) = |u_t(x,t)|^q & \text{in } \mathbb{R}^n \times \{t \geq 0\}, \\ u(x,0) = \varepsilon f_1(x), \quad u_t(x,0) = \varepsilon g_1(x) & x \in \mathbb{R}^n, \\ v(x,0) = \varepsilon f_2(x), \quad v_t(x,0) = \varepsilon g_2(x) & x \in \mathbb{R}^n \end{cases} \tag{1.1}$$

where $0 < \varepsilon \leq 1$, $1 < q,\ p < \infty$, $a \geq 1$ are constans, $\triangle$ denotes Laplace operator and $u = u(x,t)$, $v = v(x,t)$ are real unknown functions. Assume $f_i(x),\ g_i(x) \in C_0^\infty(\mathbb{R}^n)$(i=1, 2).

This work is motivated by recent results established by H. Takamura [1], Yi Zhou [2] and K. Yokoyama [3]. The system (1.1) is closely related to the scalar equation with small initial data of compact support of size of $\varepsilon$

$$u_{tt} - \triangle u = |u_t|^p, \quad (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{1.2}$$

It is known that the critical power $p_0(n)$ for the small data global existence and blowup is given by

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$$p_0(n) = \begin{cases} \dfrac{n+1}{n-1} & for \quad n \geq 2, \\ +\infty & for \quad n = 1. \end{cases}$$

For (1.2) , the conjecture was first verified by F. John [4] for the blowup part and T.C. Sideris [5] for the global existence part in the case $n = 3$ and both by J. Schaeffer [6] in the case $n = 5$. The blowup part in the case $n = 2$ was proved by J. Schaeffer [7] for $p = p_0(2)$, R. Agemi [8] proved it for $1 < p \leq p_0(2)$ by different methods from [7]. The case $n = 1$ is essentially due to K. Masuda [9] who proved the blowup results in the case $n = 1,\ 2,\ 3$ and $p = 2$. For the blowup part we have known the lifespan. For the lower bound it can be seen from F. John [4] in the case $n = 3,\ p = 2$. The upper bounds for $n = 2,\ p = 2,\ 3$ were proved by P. Godin [10]. Yi Zhou [2] studied the upper bounds in low space dimensions for all values of $0 < p \leq p_0(n)$.

For the systems of wave equations (1.1), the situation is quite similar to the scalar case if $a = 1$. For instance, when $p = q$, we can take $u = v$ and this reduces to the scalar case. Our Theorem 1 below gives a complete result in this case. The situation is quite different if $a \neq 1$. When $n = 3$ and $p = q = 2$, it is shown in [3] that the small initial data lead to global existence. When $n = 2$ and $p = q = 3$, we know from [11] that there is also global existence for small initial data. Note that blow up occurs in both cases when $a = 1$. When $n = 2$ and $p = q = 2$, the problem is still open. When $n = 1$ and $p = q = 2$, our Theorem 2 below shows that blow up occurs for small initial data.

Our main results are formulated below

**Theorem 1** *Assume $a = 1$ and*

$$\frac{(n-1)(pq-1)}{p+q+2} \leq 1, \tag{1.3}$$

$\mathrm{supp} f_i(x),\ \mathrm{supp} g_i(x) \subset \{x \in \mathbb{R}^n :\ |x| \leq k\}$,

$$\int_{\mathbb{R}^n} g_i(x)dx > 0, \tag{1.4}$$

*where $i = 1,\ 2$ and $k$ is a positive constant. Then the classical solution of (1.1) does not exist globally in $\mathbb{R}^n \times [0, \infty)$. Moreover, there exists a positive constant $C_0$ independent of $\varepsilon$, such that, the lifespan $T^*(\varepsilon)$ of the classical solution of (1.1) satisfies*

$$T^*(\varepsilon) \leq \begin{cases} C_0 \varepsilon^{-\frac{2(pq-1)}{p+q+2-(pq-1)(n-1)}} & for \quad \dfrac{(n-1)(pq-1)}{p+q+2} < 1, \\ \exp\{C_0 \varepsilon^{-\frac{2(pq-1)}{p+q+2}}\} & for \quad \dfrac{(n-1)(pq-1)}{p+q+2} = 1. \end{cases}$$