J. Partial Diff. Eqs. **18**(2005), 22–34
©International Academic Publishers Vol.18, No.1

CAUCHY PROBLEM FOR ONE-DIMENSIONAL P-LAPLACIAN EQUATION WITH POINT SOURCE*

Li Yinghua, Ke Yuanyuan and Wang Zejia
(Dept. of Math., Jilin Univ., Changchun 130012, China)
(E-mail: yinghuali@email.jlu.edu.cn keyy@email.jlu.edu.cn)
(Received Jun. 21, 2004)

Abstract We prove the existence, uniqueness and finite propagation of disturbance of continuous solutions to the Cauchy problem for one-dimensional p-Laplacian equation with point source.

Key Words p-Laplacian equation; point source; existence; uniqueness.
2000 MR Subject Classification 35K55, 35K65, 35R05.
Chinese Library Classification O175.26.

1. Introduction

This paper concerns with the Cauchy problem for the following p-Laplacian equation with point source

$$\frac{\partial u}{\partial t} = D(|Du|^{p-2}Du) + \delta(x), \qquad (x,t) \in Q, \tag{1.1}$$

$$u(x,0) = 0, \qquad x \in \mathbb{R}, \tag{1.2}$$

where $\delta(x)$ is the Dirac measure, $p \geq 2$, $D = \dfrac{\partial}{\partial x}$, $Q = \mathbb{R} \times (0,+\infty)$, $\mathbb{R} = (-\infty,+\infty)$.

The equation (1.1) is an important degenerate diffusion equation, which can be used to describe many phenomena in nature such as filtration and dynamics of biological groups and so on. During the past years, there were a tremendous amount of papers devoted to such kinds of equations without singular sources. However, as we know, the investigation about the equations with measure data is quite fewer. For the case $p = 2$, in [1] Li Huilai proved the existence of the solutions to parabolic equations with measure data, and in [2] Pang Zhiyuan, Wang Yaodong and Jiang Lishang studied the optimal control problems for semilinear diffusion equations with Dirac measure. F. Abergel, A. Decarreau and J. M. Rakotoson [3] dealt with a class of equations with measure data, and studied the existence and uniqueness of the solutions of the initial boundary

*This work is partially supported by the National Science Foundation of China, and partially supported by a Specific Foundation for Ph.D Specialities of Educational Department of China.

value problem in a bounded domain. Yuan Hongjun and Wu Gang [4] investigated the porous medium equation with Dirac measure and gave the existence of the weak solutions for Cauchy problem. In 1997, Lucio Boccardo, Andrea Dall'Aglio, Thierry Gallouët and Luigi Orsina [5] studied the initial boundary value problem for a class of nonlinear parabolic equations with measure data in general form in bounded domains.

In this paper, we deal with the Cauchy problem for one-dimensional p-Laplacian equation with point source, which is quite different from the initial boundary value problem in a bounded domain. Just as did in [5], we should first make an approximation of the Dirac measure. However, based on our approach technique, we require a C^∞–approximation rather than in L^q norm. Then we approximate the Cauchy problem by a sequence of bounded domains of the form $Q_{R,T} = (-R, R) \times (0, T)$. Finally, because of the degeneracy, we use parabolic regularization to approach the equation. Based on BV estimates, L^p–type estimates and weighted energy estimates, we establish the existence and uniqueness of continuous solutions of the problem (1.1), (1.2). Precisely, we have the following result

Theorem 1.1 *The Cauchy problem (1.1), (1.2) admits one and only one continuous solution with compact support.*

By the continuous solution, we mean the following

Definition 1.1 *A nonnegative function $u : Q \longmapsto \mathbb{R}$ is said to be a continuous solution of the Cauchy problem (1.1), (1.2), if for any $T \in (0, +\infty)$, $u \in L^\infty(Q_T) \cap L^\infty(0, T; W^{1,p}(\mathbb{R})) \cap BV(Q_T)$ and the following integral equalities*

$$-\iint_{Q_T} u\frac{\partial \varphi}{\partial t}dxdt = -\iint_{Q_T} |Du|^{p-2}DuD\varphi dxdt + \int_0^T \varphi(0,t)dt, \quad \forall \varphi \in C_0^\infty(Q_T), \tag{1.3}$$

and

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}} \psi(x)u(x,t)dxdt = 0, \qquad \forall \psi \in C_0^\infty(\mathbb{R}), \tag{1.4}$$

hold, where $Q_T = \mathbb{R} \times (0, T)$.

2. Proof of the Main Result

Just as mentioned above, to discuss the existence of continuous solutions of the problem (1.1), (1.2), we first consider the regularized problem

$$\frac{\partial u}{\partial t} = D\Big(\Big(|Du|^2 + \frac{1}{n}\Big)^{(p-2)/2} Du\Big) + \delta_\varepsilon(x), \qquad (x,t) \in Q_{R,T}, \tag{2.1}$$

$$u(x, 0) = 0, \qquad x \in (-R, R), \tag{2.2}$$