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# SEMICLASSICAL STATES OF HAMILTONIAN SYSTEM OF SCHRÖDINGER EQUATIONS WITH SUBCRITICAL AND CRITICAL NONLINEARITIES*

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Dedicated to Professor K. C. Chang on his seventieth birthday

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**Abstract** We consider the system of perturbed Schrödinger equations

$$\begin{cases} -\varepsilon^2\Delta\varphi + \alpha(x)\varphi = \beta(x)\psi + F_\psi(x,\varphi,\psi) \\ -\varepsilon^2\Delta\psi + \alpha(x)\psi = \beta(x)\varphi + F_\varphi(x,\varphi,\psi) \\ w := (\varphi,\psi) \in H^1(\mathbb{R}^N,\mathbb{R}^2) \end{cases}$$

where $N \geq 1$, $\alpha$ and $\beta$ are continuous real functions on $\mathbb{R}^N$, and $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ is of class $\mathcal{C}^1$. We assume that either $F(x,w)$ is super-quadratic and subcritical in $w \in \mathbb{R}^2$ or it is of the form $\sim \frac{1}{p}P(x)|w|^p + \frac{1}{2^*}K(x)|w|^{2^*}$ with $p \in (2, 2^*)$ and $2^* = 2N/(N-2)$, the Sobolev critical exponent, $P(x)$ and $K(x)$ are positive bounded functions. Under proper conditions we show that the system has at least one nontrivial solution $w_\varepsilon$ provided $\varepsilon \leq \mathcal{E}$; and for any $m \in \mathbb{N}$, there are $m$ pairs of solutions $w_\varepsilon$ provided that $\varepsilon \leq \mathcal{E}_m$ and that $F(x,w)$ is,in addition, even in $w$. Here $\mathcal{E}$ and $\mathcal{E}_m$ are sufficiently small positive numbers. Moreover, the energy of $w_\varepsilon$ tends to 0 as $\varepsilon \to 0$.

**Key Words** Perturbed Schrödinger equation; critical nonlinearity; multiple solutions.

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## 1. Introduction and Main Results

The goal of this paper is to study the existence and multiplicity of semiclassical solutions of the following Hamiltonian system of perturbed Schrödinger equations

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$$\begin{cases} -\varepsilon^2\Delta\varphi + \alpha(x)\varphi = \beta(x)\psi + F_\psi(x,\varphi,\psi) \\ -\varepsilon^2\Delta\psi + \alpha(x)\psi = \beta(x)\varphi + F_\varphi(x,\varphi,\psi) \\ (\varphi,\psi) \in H^1(\mathbb{R}^N,\mathbb{R}^2) \end{cases} \tag{1.1}_\varepsilon$$

where $N \geq 1$, $\alpha$ and $\beta$ are continuous real functions on $\mathbb{R}^N$, and $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ is of class $\mathcal{C}^1$.

Set

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$w = (\varphi,\psi) \quad \text{or} \quad w = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \quad \text{for } w \in \mathbb{R}^2$$

and

$$\tilde{F}(x,w) = \frac{1}{2}\beta(x)|w|^2 + F(x,w),$$

the system $(1.1)_\varepsilon$ can be rewritten in the vector form

$$-\varepsilon^2\Delta w + \alpha(x)w = \mathcal{J}\tilde{F}_w(x,w), \quad w \in H^1(\mathbb{R}^N,\mathbb{R}^2) \tag{1.2}_\varepsilon$$

This equation arises in the study of the standing wave solutions of the nonlinear Schrödinger system

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi + \gamma(x)\phi - \mathcal{J}f(x,|\phi|)\phi. \tag{1.3}$$

A standing wave solution of (1.3) is a solution of the form $\phi(x,t) = w(x)e^{-\frac{iEt}{\hbar}}$. It is clear that $\phi(x,t)$ solves (1.3) if and only if $w(x)$ solves $(1.2)_\varepsilon$ with $\alpha(x) = \gamma(x) - E, \varepsilon^2 = \frac{\hbar^2}{2m}$ and $\tilde{F}_w(x,w) = f(x,|w|)w$. $(1.2)_\varepsilon$ can be also viewed as the equation for steady state solutions of diffusion systems (see, for example, [1]).

There are many works devoted to studying the semiclassical solutions of single perturbed Schrödinger equations, see [2-14] and references therein. There are also papers devoted to investigating unperturbed (i.e., $\varepsilon = 1$) elliptic systems, see [5, 15, 16]

In this paper,we assume that continuous functions $\alpha(x)$ and $\beta(x)$ satisfy the following condition

$(A_0)$ $|\beta(x)| \leq \alpha(x)$ for all $x \in \mathbb{R}^N$, $\alpha(x_0) = \beta(x_0)$ for some $x_0$, and there is $b > 0$ such that the set $\{x \in \mathbb{R}^N : \ \alpha(x) - |\beta(x)| < b\}$ has finite Lebesgue measure.

Concerning the nonlinearities we will study two cases: subcritical and critical superlinearities.

We first consider the subcritical problem. To unify the notations,for the subcritical case,we use $G(x,w)$ instead of $F(x,w)$, and write the equation as:

$$\begin{cases} -\varepsilon^2\Delta\varphi + \alpha(x)\varphi - \beta(x)\psi = G_\psi(x,w) \\ -\varepsilon^2\Delta\psi + \alpha(x)\psi - \beta(x)\varphi = G_\varphi(x,w) \\ w = (\varphi,\psi) \in H^1(\mathbb{R}^N,\mathbb{R}^2) \end{cases} \tag{$\mathcal{P}$}_\varepsilon$$