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# APPROXIMATE CONTROLLABILITY OF THE STOKES EQUATIONS WITH BOUNDARY CONDITION ON THE PRESSURE*

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**Abstract** In this paper, some properties of three-dimensional Stokes equations with boundary condition of pressure on parts of boundary are studied. By use of these properties, approximate controllability by tangent boundary controls acting on a subboundary is studied. In addition, the controllability problem is considered when its controls act on a subdomain.

**Key Words** Stokes equation; controllability; boundary condition on the pressure.
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## 1. Introduction

Approximate controllability of the Navier-Stokes equations with homogeneous Dirichlet boundary condition on a bounded domain is an open problem when its controls act on a subdomain or subboundary. In some special cases, the controllability problems have been solved. For Euler equation, the exact boundary controllability was proved (cf. [1, 2]). In [3–5], the problem was solved for Navier-Stokes equations on two-dimensional manifold without boundary, two-dimensional domain with Navier slip boundary condition and whole domain $R^3$. The exact controllability of Galërkin approximations of Navier-Stokes equations with controls acting on a subdomain was

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proved in [6]. There are some results of local exact controllability for Navier-Stokes equations (cf. [7–11]).

In the case of Stokes equations and more general linearized equations, the approximate controllability has been solved in [12–17]. In [12,17] the problem is studied under more restrictions on the controls. But those papers, except [12] deal with the case of homogeneous Dirichlet boundary condition. In [12] negative result is obtained if unidirectional controls are parallel to the axis of cylinder on cylinder domain in $R^3$, and homogeneous Dirichlet condition on side wall and periodic boundary condition on sections are considered. When one studies local exact controllability of the Navier-Stokes equations, null controllability of the linearized Navier-Stokes equation is considered. But these all deal with homogeneous Dirichlet boundary condition and Navier slip boundary condition (cf. [7, 8, 10, 11]).

On the other hand, the existence and uniqueness of solutions are studied for the Stokes and Navier-stokes equations with boundary condition on the pressure (cf. [5,18–23]).

In this paper we consider approximate controllability of Stokes equations with boundary condition of the pressure on parts of boundary.

This paper is organized as follows. In Section 2 we consider a trace of solution and a kind of unique continuation of solutions. In Section 3, the properties of a functional connected with the control problem are studied. Owing to these properties, we prove approximate controllability by tangent boundary controls. Also, the problem is considered in the case of distributed bang-bang controls.

The following notation is used:

$\Omega$ denotes a bounded domain of $R^3$ with a boundary in $C^{1,1}$, $\Gamma_i$, $i = 1, 2, 3$, denote open subset of $\partial\Omega$ such that $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, $\Gamma_i \cap \Gamma_j = \emptyset (i \neq j)$, $\bar{\Gamma}_2 \cap \bar{\Gamma}_3 = \emptyset$, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$, $\mathrm{mes}\Gamma_1 \neq 0$, $\Gamma_2 \in C^{2,1}$. If $X$ is a space, then $\mathcal{X} = \{X\}^3$. $\mathcal{H}^1(\Omega) = \{W_2^1(\Omega)\}^3$, $\mathcal{D}(\Omega)$ is the set of functions in $C_0^\infty(\Omega)$. $V = \{u \in \mathcal{H}^1(\Omega)\colon \mathrm{div} u = 0,\ u|_{\Gamma_1} = 0, u \times n|_{\Gamma_2} = 0,$ $u \cdot n|_{\Gamma_3} = 0\}$, where $n$ denotes outward normal unit vector on the boundary. The inner product in $V$ is the same as in $\mathcal{H}^1(\Omega)$. $H$ is the closure of $V$ in $\mathcal{L}_2(\Omega)$. $< \cdot, \cdot >$ denotes duality product between a space and its dual space, $(\cdot, \cdot)$ is an inner product. $X^*$ is the dual space of a space $X$. All Sobolev spaces are real value spaces. $(\cdot, \cdot)_X$ means inner product in $\mathcal{L}_2(X)$.

## 2. Properties of Stokes Equations with Boundary Condition on the Pressure

Let us consider initial boundary value problem

$$\begin{cases} u' - \mu\Delta u + \nabla P = f, \\ \mathrm{div}\ u = 0, \\ u|_{\Gamma_1} = \psi(x,t), \quad u \times n|_{\Gamma_2} = 0, \quad P|_{\Gamma_2} = P_0(x,t), \quad u \cdot n|_{\Gamma_3} = 0, \\ \qquad (\nabla \times u) \times n|_{\Gamma_3} = 0, \\ u(0) = u_0 \in H, \end{cases} \tag{2.1}$$