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Seiberg-Witten Like Monopole Equations on $\mathbb{R}^5$

DEĞIRMENCI Nedim* and KARAPAZAR Senay

Department of Mathematics, Anadolu University, Eskisehir, Turkey.

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Abstract. We give an analogy of Seiberg-Witten monopole equations on flat Euclidian space $\mathbb{R}^5$. For this we used an irreducible representation of complex Clifford algebra $\mathbb{C}l_5$. For the curvature equation we use a kind of self-duality notion of a 2-form on $\mathbb{R}^5$ which is given in [1].

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1 Introduction

Seiberg-Witten monopole equations were defined on 4-dimensional Riemannian manifolds in 1994 by E. Witten (see [2]). These are a couple of non-linear partial differential equations on a 4-dimensional Riemannian manifold and give differential topological invariants for the underlying 4-manifold (see [3]). In recent years some generalizations of Seiberg-Witten equations to higher dimensional manifolds are studied by various authors (see [4–7]). The purpose of this article is to write down similar equations on $\mathbb{R}^5$.

2 Some basic materials

2.1 $spin^c$-structure and Dirac operator on $\mathbb{R}^n$

Definition 2.1. *The vector space of complex n-spinors is the complex vector space* $S=\mathbb{C}^{2^k}$ *with the hermitian inner product, where* $k=n/2$ *if* n *is even or* $k=(n-1)/2$ *if* n *is odd. A* $spin^c$*-structure on the Euclidean space* $\mathbb{R}^n$ *is a pair* (S,Γ) *where* $\Gamma:\mathbb{R}^n\to End(S)$ *is a linear map which satisfies*

$$\Gamma(v)^*+\Gamma(v)=0, \qquad \Gamma(v)^*\Gamma(v)=|v|^2 1$$

*Corresponding author. *Email addresses:* `ndegirmenci@anadolu.edu.tr` (N. Değirmenci), `skarapazar@anadolu.edu.tr` (S. Karapazar)

for every $v \in \mathbb{R}^n$.

From the universal property of the complex Clifford algebra $\mathbb{C}l_n$ the map Γ can be extended to an algebra homomorphism $\Gamma: \mathbb{C}l_n \to \mathrm{End}(S)$ which satisfies $\Gamma(\tilde{x}) = \Gamma(x)^*$, where $\tilde{x}$ is conjugate of x in $\mathbb{C}l_n$ and $\Gamma(x)^*$ denotes the Hermitian conjugate of $\Gamma(x)$. Let $e_1, e_2, \cdots, e_n$ be the standard basis of $\mathbb{R}^n$ and $e^1, e^2, \cdots, e^n$ be its dual. If (S,Γ) is a spin^c structure on $\mathbb{R}^n$, then we can define an action of the space of 2−forms $\Lambda^2(\mathbb{R}^n)$ on S as follows: Firstly identify $\Lambda^2(\mathbb{R}^n)$ with the spaces of second order elements of Clifford algebra $C_2(\mathbb{R}^n)$ via the map

$$\begin{aligned} \Lambda^2(\mathbb{R}^n) \quad &\to \quad C_2(\mathbb{R}^n), \\ \eta = \sum_{i<j} \eta_{ij} e^i \wedge e^j \quad &\mapsto \quad \sum_{i<j} \eta_{ij} e_i e_j. \end{aligned}$$

If we compose this map with Γ, then we obtain a map $\rho: \Lambda^2(\mathbb{R}^n) \to \mathrm{End}(S)$ by

$$\rho\Big(\sum_{i<j} \eta_{ij} e^i \wedge e^j\Big) = \sum_{i<j} \eta_{ij} \Gamma(e_i)\Gamma(e_j).$$

The map ρ extends to a map

$$\rho: \Lambda^2(\mathbb{R}^n) \otimes \mathbb{C} \to \mathrm{End}(S)$$

on the space of complex valued 2−forms. By using an i$\mathbb{R}$−valued 1−form $A \in \Omega^1(\mathbb{R}^n, \mathrm{i}\mathbb{R})$ and the Levi-Civita connection ∇ on $\mathbb{R}^n$ we can obtain a connection ∇^A on S, which is called spinor covariant derivative operator and it satisfies

$$\nabla^A_V(\Gamma(W)\Psi) = \Gamma(W)\nabla^A_V \Psi + \Gamma(\nabla_V W)\Psi,$$

where Ψ is spinor, a section of S, V and W are vector fields on $\mathbb{R}^n$. At this point we can define Dirac operator $D_A: C^\infty(\mathbb{R}^n, S) \to C^\infty(\mathbb{R}^n, S)$ by

$$D_A(\Psi) = \sum_{i=1}^{n} \Gamma(e_i) \nabla^A_{e_i}(\Psi).$$

2.2 Seiberg-Witten equations on $\mathbb{R}^4$

The following form of Seiberg-Witten equations can be found in [8,9]. The *spin^c* connection $\nabla = \nabla^A$ on $\mathbb{R}^4$ is given by

$$\nabla_j \Psi = \frac{\partial \Psi}{\partial x_j} + A_j \Psi,$$

where $A_j: \mathbb{R}^4 \longrightarrow i\mathbb{R}$ and $\Psi: \mathbb{R}^4 \longrightarrow \mathbb{C}^2$. Then the associated connection on the line bundle $L_\Gamma = \mathbb{R}^4 \times \mathbb{C}$ is the connection 1−form

$$A = \sum_{i=1}^{4} A_i \mathrm{d}x_i \in \Omega^1\left(\mathbb{R}^4, i\mathbb{R}\right),$$