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Blow-Up of Solutions for a Singular Nonlocal Viscoelastic Equation

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Abstract. We study the nonlinear one-dimensional viscoelastic nonlocal problem:

$$u_{tt}-\frac{1}{x}(xu_x)_x+\int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x \mathrm{d}s=|u|^{p-2}u,$$

with a nonlocal boundary condition. By the method given in [1, 2], we prove that there are solutions, under some conditions on the initial data, which blow up in finite time with nonpositive initial energy as well as positive initial energy. Estimates of the lifespan of blow-up solutions are also given. We improve a nonexistence result in Mesloub and Messaoudi [3].

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Key Words: Blow-up; life span; viscoelastic; nonlocal problem.

1 Introduction

In this paper we consider the initial value problem with a nonlocal boundary condition for the following nonlinear viscoelastic equation:

$$u_{tt}-\frac{1}{x}(xu_x)_x+\int_0^t g(t-s)\frac{1}{x}(xu_x(x,s))_x \mathrm{d}s=|u|^{p-2}u, \qquad \text{in } (0,a)\times(0,T), \tag{1.1}$$

$$u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \qquad x\in(0,a), \tag{1.2}$$

$$u(a,t)=0, \quad \int_0^a xu(x,t)\mathrm{d}x=0, \tag{1.3}$$

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where $a<\infty$, $T<\infty$, $p>2$, and g represents the kernel of the memory term which is specified later (see (2.4) below).

The observable phenomena in physics can be described in terms of nonlocal problems, such as problems with boundary integral conditions. For the most part, these nonlocal boundary conditions appear when the data on the body can not be measured directly, but their average values are known. For instance, in some cases, prescribing the solution u (pressure, temperature,$\cdots$) pointwise is not possible, because only the average value of the solution can be estimated along the boundary or along part of it.

Generally speaking, mathematical models with nonlocal constraints are encountered in many engineering models such as heat conduction, plasma of physics, thermoelasticity, chemical diffusion, underground water flow and certain biological process. In this direction, Cannon [4], Kamynin [5] have studied some parabolic equations by using potential method. Also Ionkin [6], Ionkin and Moiseev [7] have obtained some results regarding existence and uniqueness of solutions by using Fourier method. Moreover, Bouziani [8], Mesloub [9,10], Kartynik [11], Mesloub and Bouziani [12,13], Mesloub and Lekrine [14] investigated some parabolic models by using energy method. On the other hand, Muravei and Philinovskii [15], Mesloub and Bouziani [16], Pulkina [17] treated the hyperbolic case. Recently, Mesloub and Messaoudi [3] considered Eq. (1.1)–(1.3). The authors established a blow-up result with negative initial energy and a decay result for some initial data. However, these solutions not only blow up in finite time with negative initial energy, but also blow up in finite time with nonnegative initial energy.

In this paper we shall deal with the blow up behavior of solutions for problem (1.1)–(1.3). We derive the blow-up properties of solutions of Eqs. (1.1)–(1.3) with nonpositive initial energy as well as positive initial energy by the method given in [1,2]. In this regard, we can extend the nonexistence result to the case of positive initial energy in [3]. The content of this paper is organized as follows. In Section 2, we give some lemmas, notations and state the local existence theorem. In Section 3, we define an energy function $E(t)$ and show that it is a non-increasing function of t. Then we obtain Theorem 3.4, which gives the blow-up phenomena of solutions even for positive initial energy. Estimates for the blow-up time are also given.

2 Preliminary results

In this section, we shall give some lemmas and notations which will be used throughout this work. First, we introduce some function space used in this paper.

Let $L_x^p=L_x^p(a,\,b)$ be the weighted Banach space equipped with the norm

$$\|u\|_p=\left(\int_a^b x|u|^p\mathrm{d}x\right)^{\frac{1}{p}}.$$

In particular, as $p=2$, we denote $H=L_x^2(a,b)$ to be the weighted Hilbert space of square