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A Multiplicity Result for a Singular and Nonhomogeneous Elliptic Problem in $\mathbb{R}^n$

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Abstract. We establish sufficient conditions under which the quasilinear equation

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u)+V(x)|u|^{n-2}u=\frac{f(x,u)}{|x|^\beta}+\varepsilon h(x) \qquad \text{in } \mathbb{R}^n,$$

has at least two nontrivial weak solutions in $W^{1,n}(\mathbb{R}^n)$ when $\varepsilon>0$ is small enough, $0\le\beta<n$, V is a continuous potential, $f(x,u)$ behaves like $\exp\{\gamma|u|^{n/(n-1)}\}$ as $|u|\to\infty$ for some $\gamma>0$ and $h\not\equiv 0$ belongs to the dual space of $W^{1,n}(\mathbb{R}^n)$.

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1 Introduction and main results

Let $W^{1,n}(\mathbb{R}^n)$ be the usual Sobolev space in $\mathbb{R}^n (n\ge 2)$ with the norm

$$\|u\|_{W^{1,n}}=\left(\int_{\mathbb{R}^n}(|\nabla u|^n+|u|^n)\mathrm{d}x\right)^{1/n}.$$

In this paper, we consider the quasilinear differential equation

$$-\Delta_n u+V(x)|u|^{n-2}u=\frac{f(x,u)}{|x|^\beta}+\varepsilon h(x) \qquad \text{in } \mathbb{R}^n, \tag{1.1}$$

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where $-\Delta_n u = -\mathrm{div}(|\nabla u|^{n-2}\nabla u)$, V is a continuous potential, $h \not\equiv 0$ belongs to the dual space of $W^{1,n}(\mathbb{R}^n)$, $0 \leq \beta < n$ and $f(x,u)$ behaves like $\exp\{\gamma |u|^{n/(n-1)}\}$ as $|u| \to \infty$.

This kind of elliptic problems involving exponential critical growth has been extensively studied by many authors. To get a solution, Moser-Trudinger type inequality and critical point theory are two fundamental tools. For the homogeneous and nonsingular case, that is when $h \equiv 0$ and $\beta = 0$, the existence result in a bounded domain was obtained in [1,2]. When the domain is the whole space, the problem was studied in [3–5]. We can also consider the problem in a Riemannian manifold. For this case one can refer to [6–8] and the references therein. Because of the variational structure of this kind of equations, usually there are both minimum type and mountain-pass type solutions. A nature question is that whether these two types of solutions are different. When $n = 2$ and $\beta = 0$, do Ó, Medeiros and Severo [9] proved that these are two distinct solutions. For general dimensional case, the same authors got the result in [10]. In our paper, the nonlinearity of Eq. (1.1) becomes singular. In [11], do Ó proved that there are two distinct solutions for this singular equation when $n = 2$. Then relevant issues about the general dimensional case should be asked. Our main theorem is to give sufficient conditions under which there are still two solutions to (1.1).

To present our results, we assume the following conditions on the nonlinearity $f(x,s)$:

(H_1) There exist constants $\alpha_0, b_1, b_2 > 0$ such that for all $(x,s) \in \mathbb{R}^n \times \mathbb{R}^+$,

$$|f(x,s)| \leq b_1 s^{n-1} + b_2 \Big\{ \exp\{\alpha_0 |s|^{n/(n-1)}\} - B_{n-2}(\alpha_0, s) \Big\},$$

where

$$B_{n-2}(\alpha_0, s) = \sum_{m=0}^{n-2} (\alpha_0^m |s|^{mn/(n-1)})/m!.$$

(H_2) There exist constants $p > n$ and C_p such that

$$f(s) \geq C_p s^{p-1} \qquad \text{for all } s \geq 0,$$

where

$$C_p > \left(\frac{p-n}{p}\right)^{(p-n)/n} \left(\frac{n\alpha_0}{(n-\beta)\alpha_n}\right)^{(n-1)(p-n)/n} S_p^p,$$

$$S_p := \inf_{u \in E \setminus \{0\}} \frac{\left(\int_{\mathbb{R}^n} (|\nabla u|^n + V(x) u^n) \mathrm{d}x\right)^{1/n}}{\left(\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^\beta} \mathrm{d}x\right)^{1/p}}.$$

(H_3) There exists $\mu > n$ such that for all $x \in \mathbb{R}^n$ and $s > 0$,

$$0 < \mu F(x,s) \equiv \mu \int_0^s f(x,t) \mathrm{d}t \leq s f(x,s).$$