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A Singular Trudinger-Moser Inequality in Hyperbolic Space

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Abstract. In this paper, we establish a singular Trudinger-Moser inequality for the whole hyperbolic space $\mathbb{H}^n$:

$$\sup_{u\in W^{1,n}(\mathbb{H}^n),\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u|^n\mathrm{d}\mu\le 1}\int_{\mathbb{H}^n}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}}{\rho^\beta}\mathrm{d}\mu<\infty\Longleftrightarrow\frac{\alpha}{\alpha_n}+\frac{\beta}{n}\le 1,$$

where $\alpha>0,\beta\in[0,n)$, ρ and $d\mu$ are the distance function and volume element of $\mathbb{H}^n$ respectively.

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Key Words: Singular Trudinger-Moser inequlity; hyperbolic space.

1 Introduction

In the past forty years, Trudinger-Moser inequality has play an important role in analysis and geometry. People call it Trudinger-Moser inequality because it was first proposed by Trudinger [1] in 1967: $\exists\alpha,C>0$, s.t.

$$\sup_{u\in W_0^{1,n}(\Omega),\int_\Omega|\nabla u|^n\mathrm{d}x\le 1}\int_\Omega e^{\alpha|u|^{\frac{n}{n-1}}}\mathrm{d}x\le C|\Omega|, \tag{1.1}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω, and then improved by Moser [2] in 1971: the best constant for α is $\alpha_n=n\omega_{n-1}^{\frac{1}{n-1}}$, $\omega_{n-1}=|S^{n-1}|$. Here the best constant means that: if

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$\alpha \le \alpha_n$, then inequality (1.1) holds; if $\alpha > \alpha_n$, then there exists a sequence $\{u_k\} \subset W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u_k|^n \mathrm{d}x \le 1$, but $\int_\Omega e^{\alpha u_k^2} \mathrm{d}x \to \infty$ as $k \to \infty$. As limit case of the Sobolev embedding theorem, there is no need to say the importance of (1.1) in analysis. One more word we want to say here it that, using a similar inequality, Moser [3] solved the prescribing Gauss curvature problem on $\mathbb{R}P^2$.

Roughly speaking, the classical Trudinger-Moser inequality ((1.1) with $\alpha = \alpha_n$) has the following four kinds of generalizations:

(1) To high order derivatives, i.e., to $W_0^{m,\frac{n}{m}}(\Omega)$, this work was done by Adams [4] in 1988.

(2) To compact manifolds with or without boundary, this problem was first attempted by Aubin [5] in 1970, then studied by Cherrier [6] in 1979 and solved by Fontana [7] in 1993.

(3) To the whole Euclidean spaces, this problem was first attempted by Cao [8] in 1992, then studied by Panada [9] in 1995, do Ó [10] in 1997, Ruf [11] in 2005, Li-Ruf [12] in 2008, Adimurthi-Yang [13] in 2010, and Yang-Zhu [14] in 2013.

(4) To the whole complete noncompact manifolds, this problem was first attempted by Yang [15] in 2012 for general manifolds. When manifold is $\mathbb{H}^n$, the hyperbolic space with constant sectional curvature -1, this problem was studied by Mancini-Sandeep [16] in 2010, Adimurthi-Tintarev [17] in 2010, Battaglia [18] and Mancini [19] in 2011, Wang-Ye [20] in 2012, Tintarev [21] and Mancini-Sandeep-Tintarev [22] in 2013, and Yang-Zhu [23] in 2014.

In this paper, we will establish a singular Trudinger-Moser inequality on the whole hyperbolic space $\mathbb{H}^n$. Before stating the main result, let us review some relevant results in the past few years. In 2007, Aimurthi-Sandeep [24] first derived a singular Trudinger-Moser inequality on a bounded domain in $\mathbb{R}^n$ containing the origin, they proved

$$\int_\Omega \frac{e^{\alpha |u|^{\frac{n}{n-1}}}}{|x|^\beta} \mathrm{d}x < \infty \tag{1.2}$$

and

$$\sup_{u \in W_0^{1,n}(\Omega), \int_\Omega |\nabla u|^n \mathrm{d}x \le 1} \int_\Omega \frac{e^{\alpha |u|^{\frac{n}{n-1}}}}{|x|^\beta} \mathrm{d}x < \infty \Longleftrightarrow \frac{\alpha}{\alpha_n} + \frac{\beta}{n} \le 1, \tag{1.3}$$

where $\alpha > 0, \beta \in [0,n)$. In 2010, Adimurthi-Yang [13] generalized (1.3) to the whole Euclidean space $\mathbb{R}^n$, they obtained

$$\sup_{||u||_{1,\tau} \le 1} \int_{\mathbb{R}^n} \frac{e^{\alpha |u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{\frac{nk}{n-1}}}{k!}}{|x|^\beta} \mathrm{d}x < \infty \Longleftrightarrow \frac{\alpha}{\alpha_n} + \frac{\beta}{n} \le 1, \tag{1.4}$$

where $||u||_{1,\tau} = \left(\int_{\mathbb{R}^n} (|\nabla u|^n + \tau |u|^n) \mathrm{d}x \right)^{\frac{1}{n}}$, $\alpha > 0$ and $\beta \in [0,n)$. Then in 2012, with the help of (1.4), Yang [25] obtained some existence results of positive solutions to quasi-linear