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Existence and Properties of Radial Solutions of a Sub-linear Elliptic Equation

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Abstract. A non lipschitzian nonlinear elliptic equation is reviewed and results of existence, uniqueness, positivity and classification are proved using direct methods derived from the equation.

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1 Main results

In this paper we focus on the study of the equation

$$\Delta u+u-|u|^{-2\theta}u=0, \quad \text{on } \mathbb{R}^d, \tag{1.1}$$

with $d>1$, and $0<\theta<1/2$.

Such a problem and more generally $\Delta u+f(x,u)=0$ has been the object of numerous studies because of its interests. Indeed, it can be understood as a time-dependent problem such as

$$\begin{cases} \mathcal{L}(t,u)+f(t,u)=0, & \text{on } \Omega\times[0,T],\\ u(x,t)=0, & \text{on } \partial\Omega\times[0,T],\\ u(x,0)=u_0(x), \end{cases} \tag{1.2}$$

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where $\mathcal{L}(t,u) = i\partial u/\partial t - \Delta u$ for the famous Schrödinger operator for example, $\mathcal{L}(t,u) = \partial u/\partial t - \Delta u$ for Heat equation, $\mathcal{L}(t,u) = \partial^2 u/\partial t^2 - \Delta u$ for Wave equation, ..., and with a suitable domain $\Omega \subset \mathbb{R}^d$. The easy case is when non linear term $f(u)$ is assumed to be locally Lipschitz continuous which is not the case in many interesting cases in physics such as nonlinear waves, Shrödinger equation when dealing with complex case, chemical reaction models, population genetics problems, reactor dynamics and heat conduction. For backgrounds on (1.2), we refer to [1] and the references therein.

Problem (1.1) has been firstly studied in [2] using a shooting method and a phase plane analysis to prove the existence of nodal radial compactly supported solutions. In [3], a classification of the solutions of the problem $\Delta u + |u|^{p-1}u + \lambda u = 0$, in the unit ball B_1 in $\mathbb{R}^d$, with $d \geq 2$ and $p > 1$ have been developed. The authors analyzed in subcritical, critical and supercritical cases the possible singularities of the solution at the origin and obtained thus three different classes. The critical behavior is related to the exponent p whom critical value is $p_c = (d+2)/(d-2)$ for $d \geq 3$. In [4] and [5], the authors have focused on the mixed case $f(u) = |u|^{p-1}u + \lambda|u|^{q-1}u$ with $0 < q < 1 < p$ depending as usual on p_c. A classification of solutions has been established and nodal solutions has been proved to exist by using variational methods as well as shooting ones already emerged with ODEs.

In [6], the original famous known as Brezis-Nirenberg problem has been considered. The authors established the existence of positive solutions of $\Delta u + u^{p_c} + f(x,u) = 0$ on Ω, and $u = 0$ on $\partial\Omega$ where Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 3$ and $f(x,u)$ is a lower-order perturbation of u^{p_c} in the sense that $\lim\limits_{u\to\infty} f(u)/u^{p_c} = 0$. Next, an interesting study was developed in [7] by considering the problem $\alpha(\Delta u + u) - u^{1-\alpha} = 0$ on a ball B_R in $\mathbb{R}^d$ and $u = 0$ on ∂B_R and with a parameter $\alpha \in]1,2[$. In [8], R. Kajikiya studied (1.1) in the sub-linear case $f(s) = |s|^{p-1}$ with $0 < p < 1$. Under suitable assumptions extracted from f, the author established a sufficient and necessary condition for the existence and uniqueness of radially symmetric nodal solution. Next, a famous study of problem (1.1) has been developed in [9] where the author originally considered the problem and proved the existence of a ground state and infinitely many radial solutions which are precisely compactly supported. The author proved also that other solutions exist and these are oscillating on ± 1 with a finite number of zeros. Positive solutions which are tending to zero at infinity are necessarily compactly supported. Finally, a more complicated situation has been considered recently in [10] where the author have considered the problem

$$\Delta u + \lambda u^p - \chi_{[u>0]} u^{-\beta} = 0, \quad u \geq 0 \quad \text{on } \Omega,$$

with $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain with smooth boundary, $\lambda > 0$, $0 < \beta < 1$, $1 \leq p < p_c$. The authors proved the existence of nontrivial solutions without restrictions on λ and studied the behavior of solutions according to it. For example, as $\lambda \to \infty$, they proved that the least energy solutions concentrate around a point that maximizes the distance to the boundary.