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On Approximation by Reciprocals of Polynomials with Positive Coefficients

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Abstract. In order to study the approximation by reciprocals of polynomials with real coefficients, one always assumes that the approximated function has a fixed sign on the given interval. Sometimes, the approximated function is permitted to have finite sign changes, such as $l(l\geq 1)$ times. Zhou Songping has studied the case $l=1$ and $l\geq 2$ in L^p spaces in order of priority. In this paper, we studied the case $l\geq 2$ in Orlicz spaces by using the function extend, modified Jackson kernel, Hardy-Littlewood maximal function, Cauchy-Schwarz inequality, and obtained the Jackson type estimation.

Key Words: Approximation, polynomial, Steklov function, Orlicz space, modulus of continuity.
AMS Subject Classifications: 41A17, 41A20

1 Introduction and main result

Denote by $\Pi_n(+)$ the set of all polynomials with positive coefficients of degree n, that is

$$\Pi_n(+)=\Big\{P_n(x):P_n(x)=\sum_{0\leq k+l\leq n} a_{k,l}x^k(1-x)^l, a_{k,l}>0\Big\}.$$

In order to consider approximation by reciprocals of polynomials with real coefficients,we always assume that the given function f has a fixed sign on the given interval. In general,we allow the function f to have finite sign changes, such as $l(l\geq 1)$ times, and this result was first given by Leviatan, Lubinsky in [1]. They proved the following.

Theorem 1.1. *Let $f(x)\in C_{[-1,1]}$ change its sign exactly l times in $(-1,1)$, say at $-1<b_1<b_2<\cdots<b_l<1$, then for each $n\geq 1$, there exists $P_n(x)\in\Pi_n(+)$ having the same sign as f in $(b_l,1)$, such that for $x\in[-1,1]$*

$$\Big\|f(x)-\frac{\prod_{j=1}^{l}(x-b_j)}{p_n(x)}\Big\|_C\leq C(l+1)^2\omega\Big(f,\frac{1}{n}\Big)_C.$$

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In [2], Zhou partly generalized the result in [1] for the case $l=1$, $1<p<\infty$. In [3], Wang and Wu generalized the result in [2] to Orlicz spaces. In [4], Zhou and Mei studied the case $f(x)\in L^p_{[0,1]}$ $(1<p<\infty)$ and have sign changes $l(l\geq 2)$ times, they obtained

Theorem 1.2. *Let $f(x)\in L^p_{[0,1]}$ $(1<p<\infty)$, and change sign exactly $l(l\geq 2)$ times in $(0,1)$, then there exist $0<b_1<b_2<\cdots<b_l<1$ and $P_n(x)\in\Pi_n(+)$, such that*

$$\left\|f(x)-\frac{\prod_{j=1}^{l}(x-b_j)}{p_n(x)}\right\|_{L^p_{[0,1]}}\leq C_{p,b,l}\,\omega(f,n^{-\frac{1}{2}})_{L^p_{[0,1]}},$$

where $b=\min\{|b_{j+1}-b_j|: j=1,2,\cdots,l-1\}$, $C_{p,b,l}$ is a positive constant depending only on p, b and l.

In this paper we consider the similar problem in Orlicz spaces.

Let $M(u)$ and $N(v)$ be mutually complementary N functions, the definition and properties of N function can be seen in [5]. The Orlicz space $L^*_{M(G)}$ corresponding to the N function $M(u)$ consists of all Lebesgue measurable functions $u(x)$ on G, of which the Orlicz norm

$$\|u\|_M=\sup_{\rho(v,N)\leq 1}\left|\int_G u(x)v(x)\mathrm{d}x\right| \tag{1.1}$$

is finite, here

$$\rho(v,N)=\int_G N(v(x))\mathrm{d}x$$

is the modulus of $v(x)$ with respect to $N(v)$. According to [5], the Orlicz norm (1.1) can also be calculated by

$$\|u\|_M=\inf_{\alpha>0}\frac{1}{\alpha}\left(1+\int_G M(\alpha u(x))\mathrm{d}x\right). \tag{1.2}$$

Define the modulus of smoothness of the function $f(x)\in L^*_{M(G)}$ as

$$\omega(f,t)_M=\sup_{0\leq h\leq t}\|f(\cdot+h)-f(\cdot)\|_{M(I_h)},$$

where $I_h=[0,1-h]$ and $0\leq t<1$.

Definition 1.1. Let $f(x)\in L^*_{M[0,1]}$, we say $f(x)$ changes its sign exactly l times at $a_1,a_2,\cdots,a_l$, if there exist l points $0<a_1<a_2<\cdots<a_l<1$, such that

$$\sigma\,\mathrm{sgn}\Big(\prod_{j=1}^{l}(x-a_j)\Big)f(x)>0\quad a.e.\quad x\in[0,1],\quad \sigma=\pm 1,$$

and such that for every $j=1,2,\cdots,l$, any $0<\eta<a_{j+1}-a_j$ $(a_{l+1}=1)$,

$$\mathrm{meas}\big(\{x\in(a_j,a_{j+1}): f(x)\neq 0\}\cap(a_j,a_{j+\eta})\big)>0,$$

where we require $\mathrm{meas}\{x\in[0,a_1]: f(x)\neq 0\}>0$.