Analysis in Theory and Applications
Anal. Theory Appl., Vol. **30**, No. 1 (2014), pp. 20-33

DOI: 10.4208/ata.2014.v30.n1.2

Global Meromorphic Solutions of Partial Differential Equations

Pei-Chu Hu*

School of Mathematics, Shandong University, Jinan 250100, China

Received 9 October 2013; Accepted (in revised version) 3 March 2014

Available online 31 March 2014

Abstract. In this paper, we survey some recent progress on analytic theory of partial differential equations.

Key Words: Analytic theory, global meromorphic solutions, partial differential equations.

AMS Subject Classifications: 35A05, 35A20, 35B05, 35C10 32A05, 32A10, 32A20, 32A22

1 Introduction

We [19] studied meromorphic solutions of homogeneous linear partial differential equations of the second order in two independent complex variables

$$a_0\frac{\partial^2 u}{\partial t^2}+2a_1\frac{\partial^2 u}{\partial t\partial z}+a_2\frac{\partial^2 u}{\partial z^2}+a_3\frac{\partial u}{\partial t}+a_4\frac{\partial u}{\partial z}+a_6u=0, \tag{1.1}$$

where $a_k=a_k(t,z)$ are holomorphic functions for $(t,z)\in\Sigma$, where Σ is a region on $\mathbb{C}^2$.

When t and z are real variables, Hilbert's 19-th problem conjectures that if all $a_k=a_k(t,z)$ are analytic on t and z, then any solution $u=u(t,z)$ of an elliptic equation of the form (1.1) also is analytic on its existing region, which was confirmed by S. N. Bernšteĭn [3] provided one knows that $u\in C^3$.

H. Lewy [23], using the solvability of the initial value problem for hyperbolic equations, gave a simple proof by extending t and z to a domain of $\mathbb{C}^2$. Further, I. G. Petrovskiĭ [29] extended this result to general non-linear elliptic systems. It also is known that all regular solutions of linear elliptic equations of the second order have bounded derivatives up to order k, provided all coefficients have bounded derivatives up to order k.

We follow Lewy's idea to study the Eq. (1.1) on a region $\Sigma\subseteq\mathbb{C}^2$.

*Corresponding author. *Email address:* pchu@sdu.edu.cn (P. C. Hu)

2 Linear partial differential equations of second order related to Bessel functions

To explain our idea clearly, here we examine the following special differential equation:

$$t^2\frac{\partial^2 u}{\partial t^2}-z^2\frac{\partial^2 u}{\partial z^2}+t\frac{\partial u}{\partial t}-z\frac{\partial u}{\partial z}+t^2u=0. \tag{2.1}$$

Theorem 2.1. *The differential equation* (2.1) *has an entire solution* $f(t,z)$ *on* $\mathbb{C}^2$ *if and only if* f *is an entire function expressed by the series*

$$f(t,z)=\sum_{n=0}^{\infty} n!c_nJ_n(t)z^n, \tag{2.2}$$

such that

$$\limsup_{n\to\infty}\sqrt[n]{|c_n|}=0, \tag{2.3}$$

where $J_n(t)$ *is the first kind of Bessel's function of order* n. *Moreover, the order* $\mathrm{ord}(f)$ *of the entire function* f *satisfies*

$$\rho\le \mathrm{ord}(f)\le \max\{1,\rho\},$$

where

$$\rho=\limsup_{n\to\infty}\frac{2\log n}{\log\frac{1}{\sqrt[n]{|c_n|}}}. \tag{2.4}$$

By definition, the order of f is defined by

$$\mathrm{ord}(f)=\limsup_{r\to\infty}\frac{\log^+\log^+M(r,f)}{\log r},$$

where

$$\log^+x=\begin{cases}\log x, & \text{if } x\ge 1,\\ 0, & \text{if } x<1,\end{cases}$$

and

$$M(r,f)=\max_{|t|\le r,|z|\le r}|f(t,z)|.$$

G. Valiron [34] showed that each transcendental entire solution of a homogeneous linear ordinary differential equation with polynomial coefficients is of finite positive order. However, Theorem 2.1 shows that Valiron's theorem is not true for general partial differential equations. Here we exhibit another example that the following equation

$$t^2\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial z^2}+t\frac{\partial u}{\partial t}=0$$

has an entire solution $\exp(te^z)$ of infinite order.