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NUMERICAL ANALYSIS OF A THERMOELASTIC DIFFUSION PROBLEM WITH VOIDS

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Abstract. In this paper we consider, from the numerical point of view, a thermoelastic diffusion porous problem. This is written as a coupled system of two hyperbolic equations, for the displacement and porosity fields, and two parabolic equations, for the temperature and chemical potential fields. Its variational formulation leads to a coupled system of four parabolic variational equations in terms of the velocity, porosity speed, temperature and chemical potential. The existence and uniqueness of weak solutions, as well as an energy decay property, are recalled. Then, the numerical approximation is introduced by using the finite element method for the spatial approximation and the implicit Euler scheme to discretize the time derivatives. A stability property is proved and some a priori error estimates are obtained, from which the convergence of the algorithm is derived and, under suitable additional regularity conditions, its linear convergence is deduced. Finally, some numerical approximations are presented to demonstrate the accuracy of the algorithm and to show the behaviour of the solution.

Key words. Thermoelasticity, porosity, diffusion, finite element approximations, error estimates and numerical simulations.

1. Introduction

The field of diffusion in solids has been a topic with an increasing interest, after the second world war, in the development of high technologies. In this way, the problems dealing with the diffusion of matter in thermoelastic bodies, and the interaction of mechano-diffusion processes, have become the subject of research of many authors since the first thermoelastic diffusion theory introduced by Nowacki ([19]). Nowadays, thermodiffusion problems are very interesting to oil companies for more efficient extraction of oil from oil deposits. In such problems, diffusion can be defined as the random walk, of an ensemble of particles, from regions of high concentrations to regions of lower concentrations ([1]).

There is a number of theories about mechanical properties of granular materials. Here, we follow the theory developed by Goodman and Cowin (see [14]), who established that the mass material admits a decomposition into the density of the matrix material and the volume fraction field (the porosity), introducing a new kinematic variable. Then, few years later Nunziato and Cowin proposed in [20] a new theory to describe the properties of homogeneous elastic materials with voids free of fluid. This has been shown to be useful for the study of rocks, soils and manufactured porous materials as ceramics and pressed powders. Moreover, Ieşan developed a linear theory of thermoelastic materials with voids generalizing some ideas of [13] (see, e.g., [15] or the recent monograph [16]).

In this paper, we consider, from the numerical point of view, a thermoelastic diffusion problem with voids presented in [1, 4]. The linear thermoelasticity theory with voids is extended to include diffusion effects, leading to a new and interesting problem. In this case, thermodiffusion is due to the coupling of the fields of temperature, porosity, mass diffusion and that of strain. Related problems have been considered by other authors in, for instance, [9, 10, 17, 21] (without the diffusion effect) or [2, 3, 5, 6]. Here, we concentrate on

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the numerical approximation of this problem, introducing fully discrete approximations by using the finite element method and the implicit Euler scheme, proving a stability property and some a priori error estimates, and providing some numerical simulations.

The outline of this paper is as follows. In Section 2, the mathematical model and its variational formulation are presented. An existence and uniqueness result and an energy decay property, proved in [4], are recalled. Then, fully discrete approximations are introduced in Section 3 by using the finite element method for the spatial approximation and the backward Euler scheme for the discretization of the time derivatives. A priori error estimates are obtained, from which the convergence of the algorithm is derived and its linear convergence is deduced under suitable additional regularity conditions. Finally, in Section 4 some numerical examples are shown to demonstrate the accuracy of the algorithm and the behaviour of the solution.

2. The model and its variational formulation

Let us denote by $[0,\ell]$, $\ell > 0$, and $[0,T]$, $T > 0$, the one-dimensional rod of length ℓ and the time interval of interest, respectively. Moreover, let $x \in [0,\ell]$ and $t \in [0,T]$ be the spatial and time variables. In order to simplify the writing, we do not indicate the dependence of the functions on x and t, and the subscript under a variable represents its derivative with respect to the prescribed variable.

According to [4], a porous themoelastic rod with diffusion is considered assuming that the matrix material is elastic and that interstices are void of material, constituting a generalization of the classical theory of elasticity (see, for instance, [12, 13]). Therefore, let u, ϕ, θ, $P \in \mathbb{R}$ be the displacement field of the solid elastic material, the porosity (or volume fraction), the difference of the temperature between the actual state and a reference temperature, and the chemical potential, respectively.

Therefore, following [1, 4] the mechanical problem of a one-dimensional porous thermoelastic rod with diffusion is written as follows.

Problem P. *Find the displacement field* $u : [0,\ell] \times [0,T] \to \mathbb{R}$*, the porosity field* $\phi : [0,\ell] \times [0,T] \to \mathbb{R}$*, the temperature field* $\theta : [0,\ell] \times [0,T] \to \mathbb{R}$ *and the chemical potential field* $P : [0,\ell] \times [0,T] \to \mathbb{R}$ *such that*

$$\rho u_{tt} = \alpha u_{xx} + b\phi_x - \gamma_1 \theta_x - \gamma_2 P_x \quad \text{in} \quad (0,\ell) \times (0,T), \tag{1}$$

$$J\phi_{tt} = \eta \phi_{xx} - b u_x - \delta \phi + m_1 \theta + m_2 P \quad \text{in} \quad (0,\ell) \times (0,T), \tag{2}$$

$$c\theta_t = k^* \theta_{xx} - \gamma_1 u_{tx} - m_1 \phi_t - \kappa P_t \quad \text{in} \quad (0,\ell) \times (0,T), \tag{3}$$

$$\nu P_t = h^* P_{xx} - \gamma_2 u_{tx} - m_2 \phi_t - \kappa \theta_t \quad \text{in} \quad (0,\ell) \times (0,T), \tag{4}$$

$$u(0,t) = u(\ell,t) = 0, \quad \phi(0,t) = \phi(\ell,t) = 0, \quad \theta(0,t) = \theta(\ell,t) = 0,$$

$$P(0,t) = P(\ell,t) = 0 \quad \text{for a.e.} \quad t \in (0,T), \tag{5}$$

$$u(x,0) = u_0(x), \quad \phi(x,0) = \phi_0(x), \quad \theta(x,0) = \theta_0(x) \quad \text{for a.e. } x \in (0,\ell), \tag{6}$$

$$u_t(x,0) = v_0(x), \quad \phi_t(x,0) = e_0(x), \quad P(x,0) = P_0(x) \quad \text{for a.e. } x \in (0,\ell). \tag{7}$$

In this system of equations, ρ is the mass density, α is the elastic coefficient, γ_1, γ_2 represent thermal expansion coefficients, b is a porosity coefficient, $J = \rho\, k$, where k is the equilibrated inertia, m_1, m_2 are thermal expansion coefficients, η, δ represent porosity diffusion coefficients, κ is a thermal expansion coefficient, c is the heat capacity, ν denotes the diffusion relaxation time, k^* represents a thermal diffusion coefficient, h^* is a diffusion coefficient, and u_0, v_0, ϕ_0, e_0, θ_0 and P_0 are given initial conditions. Moreover, homogeneous Dirichlet boundary conditions have been assumed for the sake of simplicity for all the variables, but