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A HIGH PHYSICAL ACCURACY METHOD FOR INCOMPRESSIBLE MAGNETOHYDRODYNAMICS

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Abstract. We present an energy, cross-helicity and magnetic helicity preserving method for solving incompressible magnetohydrodynamic equations with strong enforcement of solenoidal constraints. The method is a semi-implicit Galerkin finite element discretization, that enforces pointwise solenoidal constraints by employing the Scott-Vogelius finite elements. We prove the unconditional stability of the method and the optimal convergence rate. We also perform several numerical tests verifying the effectiveness of our scheme and, in particular, its clear advantage over using the Taylor-Hood finite elements.

Key words. MHD, Cross-helicity, Magnetic-helicity, Scott-Vogelius elements.

1. Introduction

The conservation equations for incompressible magnetohydrodynamic (MHD) flows describe conducting, non-magnetic fluids, such as salt water, liquid metals, plasmas and strong electrolytes [7]. We will study finite element discretizations of the MHD equations in the following form, originally developed by Ladyzhenskaya, and studied in, e.g., [10, 18, 11, 12, 17]:

$$u_t + \nabla \cdot (uu^T) - Re^{-1}\Delta u + \frac{s}{2}\nabla(B\cdot B) - s\nabla\cdot BB^T + \nabla p = f, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$B_t + Re_m^{-1}\nabla\times(\nabla\times B) + \nabla\times(B\times u) = \nabla\times g, \tag{1.3}$$

$$\nabla\cdot B = 0. \tag{1.4}$$

Here, u is velocity, p is pressure, f is body force, $\nabla\times g$ is a forcing on the magnetic field B, Re is the Reynolds number, Re_m is the magnetic Reynolds number, and s is the coupling number.

We study a semi-implicit Galerkin finite element discretization of (1.1)-(1.4) which enforces pointwise solenoidal velocity and magnetic fields, as well as global conservation of energy and cross-helicity; by global conservation we mean the quantities are unchanged for ideal MHD with periodic boundary conditions, and in the viscous/resistance case and more general boundary conditions the quantities are exactly balanced, analogous to the continuous case. We also prove the exact conservation of magnetic helicity for the ideal MHD system, thus showing that our model preserves all three physical quantities that are conserved in the ideal MHD. In addition to proving these conservation laws, we also prove the scheme is unconditionally stable, well-posed, and optimally convergent. Lastly, several numerical experiments are given that demonstrate the effectiveness of the scheme.

Most schemes for fluid flow simulation conserve energy, but other fundamental conservation laws are often ignored or not strongly enforced. However, when these

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other laws are correctly accounted for in the numerical scheme, resulting solutions have greater *physical accuracy*, which leads to longer time stability and accuracy. For example, Arakawa's scheme for the 2D Navier-Stokes equations(NSE) that conserves energy and enstrophy[1], Arakawa and Lamb's scheme for the shallow water equations that conserves energy and potential enstrophy [2] and those of Navon [19, 20], J.G. Liu and W. Wang's finite difference schemes for 3D axi-symmetric NSE flow that conserves energy and helicity [17] and MHD flows with symmetry that conserve energy and cross-helicity, and most recently a scheme for full 3D NSE that conserves energy and helicity [22, 6], all exhibit better long time behavior than comparable schemes that conserve only energy. The discretization we formulate and study herein for (1.1)-(1.4) is a finite element scheme that conserves all three fundamental quantities for general MHD flows - energy, cross-helicity and magnetic helicity, and is therefore also expected to exhibit good accuracy.

In addition to integral invariants, there are other conservation laws fundamental to the system (1.1)-(1.4), which are explicitly part of the continuous system as equations (1.2) and (1.4). Finite element discretizations typically enforce these laws weakly, however in MHD this is typically not sufficient. The problems that can arise from a poor enforcement of $\nabla \cdot u = 0$ are well known even for the simpler problems such as steady Navier-Stokes equations (NSE), see e.g. [15], and thus in the MHD system such physically inconsistent effects can be magnified. The requirement that $\nabla \cdot B = 0$ comes from the fact that B is derived as the curl of a electric field, and since $\operatorname{div}\operatorname{curl} = 0$ is a formal mathematical identity, for B not to be divergence free is a mathematical inconsistency. This is well known in the MHD community, and algorithms that preserve incompressibility of B provided an incompressible initial condition is given exist [18], and for those that do not, techniques such as 'divergence cleansing' can be applied to recover mathematically plausible solutions [8]. The problem of satisfying the divergence-free condition for the magnetic field is crucial in many of the MHD applications; for instance, different numerical techniques have been used to prevent the incorrect shock capturing because of the violation of $\nabla \cdot B = 0$ condition. One can find the description of such techniques in [26] and references therein. Our scheme strongly enforces (pointwise!) the solenoidal constraints by coupling the discrete analog of (1.4) to (1.3) through the addition of a corresponding Lagrange multiplier λ to (1.3), then using the Scott-Vogelius element pair to approximate both (u, p) and (B, λ) [23, 24]. Under mild restrictions, this element pair has recently been shown to be LBB stable and admit optimal approximation properties, and also implicitly enforces strong divergence free constraints when only weak enforcement is implemented [27, 21]. It has since been successfully used with the steady and time-dependent NSE [4, 16, 15, 5], and thus the extension to using it for MHD is a next natural step.

There are two natural extensions of the scheme given in Section 3 for which our analysis is relevant. The first is for a linearization of the scheme via the method of Baker [3], by linearly extrapolating the first term of the each of the four trilinear terms. All of the theory proven for the full nonlinear (Crank-Nicolson) scheme is still valid, although the convergence proof would have additional technical details. This scheme offers a significant increase in efficiency, since only one linear solve is needed at each timestep; all of our numerical experiments will employ this linearization. The second extension is for Taylor-Hood elements, provided the trilinear terms are all skew-symmetrized. Here, the global conservation of energy and cross helicity still hold as does optimal convergence, however with this element choice,