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A SIMPLE PROOF OF THE COMPLETE CONSENSUS OF DISCRETE-TIME DYNAMICAL NETWORKS WITH TIME-VARYING COUPLINGS

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Abstract. We discuss the complete consensus problem of the discrete-time dynamical networks with time-varying couplings, and provide a simple *analytic* proof for the emergence of asymptotic complete consensus. Our approach is based on the "energy estimate argument" and connectivity of the communication topology. As direct application of our main results, we obtain asymptotic complete consensus for the discrete-time Kuramoto model with local communication topology.

Key words. Complete consensus, network, time-varying coupling

1. Introduction

Consensus problem as a dynamic feature of complex networks is an active recent subject in many different disciplines such as computer sciences, statistical physics, mathematics, biology, communications and control theory, etc. due to its engineering applications in the formation controls of robots, unmanned aerial vehicles and sensor networks [4], [7]. Complete consensus means a status reaching an agreement regarding certain information of interest that depends on the state of all agents. Consensus algorithm is a dynamic interaction rule regulating the mutual information exchange between agents. In reality, information can be exchanged through direct communication or sensing. Communication link between agents is changeable due to the failure of sensing, and range limitations. Hence we need to consider the dynamically changing communication topology for real applications, e.g. nonlinear interactions between consensus dynamics and dynamically changing network structures in biological networks. In this paper, we consider the following consensus algorithm:

$$\omega_i(t+h) = \omega_i(t) + \frac{\lambda h}{N}\sum_{j=1}^{N} c_{ji}(t)\mathcal{F}(\omega_j(t)-\omega_i(t)), \qquad 1 \le i \le N, \tag{1.1}$$

where ω_i is the information of i-th agent, t is a discrete time $h, 2h, \cdots$ and $\mathcal{F}$ is the state coupling function denoting the interaction rule between agents. The time-varying network structure is monitored by the communication matrix $C(t) := (c_{ij}(t))$.

We next briefly review the related theoretical works on the consensus problem for networks with time-varying topologies; Tsitsiklis-Bertsekas-Athans [12] developed a pioneering work on the distributed computation over networks in computer science, and Jadbabaie et al [3] provide a theoretical explanation for the convergence to the Viscek type heading alignment model [13] with time-varying topologies in the realm of flocking context. This Jadbabaie's seminal work has been further generalized in several following literature, for instance, for the undirected information flow, Fax and Murray [2], Olfati-Saber and Murray [10], whereas for a directed information

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flow, Moreau [6], Ren and Beard [8] and Tanner-Jadbabaie-Pappas [11], Fang and Antsaklis [1], etc. In these previous literature, the consensus analysis is mostly based on "*algebraic method*" such as the matrix theory and spectral graph theory to estimate the second eigenvalue(algebraic connectivity) (see [9] for a detailed review).

The purpose of this paper is to present a simple elementary approach for the asymptotic complete consensus based on the energy type estimates. Our proposed approach do not use any explicit spectral information on the eigenvalues. Instead, it is mainly dependent on the elementary inequalities and energy production rates resulting from the basic energy estimates (see Theorem 3.1 and 3.2 in Section 3).

This paper is divided into four sections after this introduction. In Section 2, we present a framework for the asymptotic consensus and several a priori estimates. In Section 3, we present a rigorous complete consensus estimate for the proposed consensus model. In Section 4, we apply main results in Section 3 to the discrete-time Kuramoto model. Finally Section 5 is devoted to the summary of main results, comparison with previous literature and future directions.

2. Preliminaries

In this section, we provide a framework on the complete consensus, and present several a priori estimates for the discrete-time system (1.1).

Let ω_i be the information state of i-th agent whose continuous-time dynamics is governed by the system with time-dependent communications:

$$\frac{d\omega_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} c_{ji}(t)\mathcal{F}(\omega_j - \omega_i), \quad 1 \le i \le N, \tag{2.1}$$

where λ is a positive coupling constant and $c_{ji} = c_{ji}(t)$ is a nonnegative function denoting the communication weight carried from j-th agent to i-th agent, moreover $\mathcal{F}$ is an odd coupling function. The standard discretization procedure for (2.1) reduces to the discrete-time dynamical model (1.1). To relate the communication topology with energy estimates, we associate the system (1.1) with the dynamic graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ at $t = nh \ge 0$:

$$\begin{aligned} &\mathcal{V} \;:\; \text{the set of all nodes,} \\ &\mathcal{E}(t) \;:\; \text{the set of all pairs } (i,j) \in \mathcal{V}\times\mathcal{V} \text{ with } c_{ij} > 0. \end{aligned}$$

2.1. A framework for complete consensus. In this part, we list main assumptions on the communication topology and the coupling function:

- ($\mathcal{H}1$) The switching communication topology $C(t) = (c_{ij}(t)), t = nh$ is symmetric and bounded:

$$c_{ij}(t) = c_{ji}(t) \le C_u < \infty, \quad \forall\, i,j,t, \qquad C_l := \inf_{i,j,t}\{c_{ij}(t) \;:\; c_{ij}(t) > 0\} > 0.$$

- ($\mathcal{H}2$) The accumulative switching communication topologies contains infinitely often completely connected paths in the sense that for some divergent sequence $\{T_i\}_{i=1}^{\infty}$,

$$1 \le T_1 < T_2 < \cdots < T_n \to \infty,$$

$$\cup_{j=T_i}^{T_{i+1}} \mathcal{E}(j) \quad \text{contains a connected path,}$$

  i.e., every two nodes in $\mathcal{V}$ can be reachable from each other in the time interval $[T_i, T_{i+1})$.