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ON THE ORDER OF OPERATOR SPLITTING METHODS FOR TIME-DEPENDENT LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. One way to solve complicated systems of differential equations is the application of operator splitting techniques. The original problem is split into several subsystems that are solved cyclically one after the other. Naturally, this procedure introduces an error, which is called splitting error, into the calculations. It is known that if the splitting procedure is applied to autonomous systems of ordinary differential equations, then the frequently used splitting procedures: the sequential, the Strang-Marchuk and the symmetrically weighted sequential splittings generally have splitting errors of order one and two, respectively. In this paper, we show that the order of the splitting procedures is preserved for non-autonomous problems. The theoretical results will be verified on numerical test problems.

Key words. Splitting methods, splitting order, Magnus expansion, non-autonomous systems

Consider the Cauchy problem (CP)

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = A(t)y + g(t), \quad t \in (0,T] \\ y(0) = y_0, \end{cases} \tag{0.1}$$

where $y_0 \in \mathbb{R}^d$, and $A(t) \in \mathbb{R}^{d\times d}$ and $g(t) \in \mathbb{R}^d$ are time-dependent, sufficiently smooth matrix functions, which guarantee the required smoothness of the solution. This problem appears in many branches of mathematics and physics, such as quantum mechanics, Hamiltonian dynamics, dynamical systems, stability analysis, semi-discretization of partial differential equations, etc. Despite its apparent simplicity, the Cauchy problem is generally solved numerically. One part of the solution methods is based on operator splitting techniques.

Splitting methods are very useful techniques for the numerical solution of complicated time-dependent real-life problems. The method was successfully applied e.g. to air-pollution models and to the numerical treatment of Maxwell equations (e.g. [8, 12, 14]). The basic idea of the method is to split the original problem (0.1) into two subproblems as

$$\frac{\mathrm{d}y^{(1)}}{\mathrm{d}t} = B(t)y^{(1)} + g^{(B)}(t) \qquad \frac{\mathrm{d}y^{(2)}}{\mathrm{d}t} = C(t)y^{(2)} + g^{(C)}(t) \quad t \in (0,\tau], \tag{0.2}$$

where $A(t) = B(t) + C(t)$, $g(t) = g^{(B)}(t) + g^{(C)}(t)$, and then to solve these systems cyclically on successive intervals all with length $\tau > 0$ using the solution of one system as the initial condition of the other one. The step size τ is called splitting timestep. The most frequently applied splitting methods are the sequential splitting [2], the Strang-Marchuk (SM) splitting [17, 19] and the symmetrically weighted sequential (SWS) splitting [7, 10]. The splitting procedure introduces a so-called splitting error. The local splitting error is defined as $E_{\mathrm{spl}}(\tau) := y(\tau) - y_{\mathrm{spl}}(\tau)$, where y_{spl} is the approximation of the exact solution at the point τ obtained by the splitting procedure. A splitting method is said to have accuracy of order p if $E_{\mathrm{spl}}(\tau) = \mathcal{O}(\tau^{p+1})$.

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This order p usually defines the convergence order of the corresponding splitting method. It is well known that the sequential, the SM and the SWS splitting methods have, respectively, first, second and second order accuracy when they are applied to autonomous linear homogeneous systems ($A(t) \equiv A \in \mathbb{R}^{d\times d}$, $g(t) \equiv 0$). It was proven in [8] that the methods preserve their order of accuracy also for autonomous linear inhomogeneous systems ($A(t) \equiv A \in \mathbb{R}^{d\times d}$) even when the inhomogeneous term depends on the independent variable t.

We remark that the split subsystems are generally solved numerically, thus the splitting methods are combined with numerical methods. The order of the combined method is the minimum of the order of the splitting method and the order of the numerical schemes applied to the subsystems [11].

The qualitative properties of non-autonomous systems and the reliability of numerical schemes are investigated in different works [1]. In [4] the convergence of various operator splitting procedures (sequential, Strang and weighted splitting) is investigated for autonomous Cauchy problems, in the presence of spatial approximation. In our paper we extend these results on the splitting error to time-dependent linear problems. Operator splitting for non-autonomous problems have been investigated in [3], too, where the main technical tool is the application of evolution semigroup methods.

Some splitting methods are already in use in the numerical solutions of different special non-autonomous systems, particularly Hamiltonian ones. In papers [5, 6] the authors apply the usual splitting methods for partitioned systems. With this technique they can get rid of the expensively computable commutators and can define higher order efficient schemes. In [9], the original matrix function $A(t)$ is split into the sum of a lower triangular and a strictly upper triangular matrix and the subsystems are solved by cyclic quadrature.

In some cases, such as the Hamiltonian dynamics or Schrödinger's equation in quantum mechanics, it is important to construct such numerical schemes that preserve some important qualitative properties and geometric structures of the solutions. These types of methods are called geometric integrators [13], which are more stable and more accurate than classical methods for certain problems.

Popular geometric integration methods for (0.1) are based on the so-called Magnus expansion [15, 16]. We will use this method in the solutions of the split subsystems in the estimations of the splitting error. The basic idea of the method is to express the fundamental solution $Y(t)$ of (0.1) in the form

$$Y(t) = \exp(\Omega(t)),$$

where the exponent $\Omega(t)$ is an infinite sum of nested integrals and commutators $\Omega(t) = \Omega_1(t) + \Omega_2(t) + \dots$ with

$$\Omega_1(t) = \int_0^t A(s)\mathrm{d}s,\ \Omega_2(t) = \frac{1}{2}\int_0^t \int_0^{s_1} [A(s_1), A(s_2)]\,\mathrm{d}s_2\,\mathrm{d}s_1,\ \dots. \tag{0.3}$$

It is known that if t is sufficiently small, namely if t satisfies the inequality

$$\int_0^t \|A(s)\|\,\mathrm{d}s < \pi,$$

then the series is absolutely convergent provided that $A(t)$ is bounded [18]. The solution of (0.1) can be written in the form

$$y(t) = Y(t)\left(\int_0^t Y^{-1}(s)g(s)\,\mathrm{d}s + y_0\right),$$