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A MESH ADAPTATION METHOD FOR 1D-BOUNDARY LAYER PROBLEMS

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Abstract. We present a one-dimensional version of a general mesh adaptation technique developed in [1, 2] which is valid for two and three-dimensional problems. The simplicity of the one-dimensional case allows to detail all the necessary steps with very simple computations. We show how the error can be estimated on a piecewise finite element of degree k and how this information can be used to modify the grid using local mesh operations: element division, node elimination and node displacement. Finally, we apply the whole strategy to many challenging singularly perturbed boundary value problems where the one-dimensional setting allows to push the adaptation method to its limits.

Key words. hierarchical error estimation, mesh adaptation, singular perturbation, boundary layer problems.

1. Introduction

Adaptive numerical methods have now a long history in efficiently computing approximations to ordinary or partial differential equations. Their goal is to reach a given level of precision at a minimal cost which often means a minimal number of degrees of freedom (DOFs). This also results in minimizing the size of the resulting algebraic systems of equations to be solved. In the case of partial differential equations (PDEs) and the finite element method – one of the most important approximation methods for PDEs – the two key ingredients are an *a posteriori* error estimator and a mesh adaptation procedure. The adaptation method uses the information from the error estimator to modify the mesh in order to reach the desired error level. For each of these two steps, there exist a large variety of possibilities. Our goal here is not to review all or even several of these methods and we refer the interested reader to [3, 4, 5, 6, 7] for starting points.

In this paper, we present a one-dimensional version of a general technique developed in [1, 2] for two and three-dimensional problems. The presented method does not depend on the PDE itself (unlike some other error estimations, based on residuals, as in [8]) and can be applied to finite element solutions of any degree.

The adaptation method is based on an error estimator that can be easily calculated on each element. As we shall see, our error estimator has a nice interpretation linking the finite element error to the classical Lagrange interpolation error. Once the error has been estimated, local operations are used to modify the mesh: element division, node elimination and node displacement. This indirectly means that the node positions are also unknown and must therefore be determined through an iterative method.

The first main goal of the present paper is to take advantage of the one-dimensional setting to give all the details of the adaptive strategy: the explicit construction of

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the error estimator, the computation of the local errors on a patch of adjacent elements and the decision process for the local mesh modifying operations. This is done for two-point boundary value problems.

The second main goal is to illustrate the efficiency of the method through a variety of problems, some of them having proved to be particularly challenging steady convection-diffusion-reaction equations. When the diffusion coefficient is small compared to the convection velocity, solutions present steep variations localized in so-called *boundary layers*. Finite element approximations then present unphysical oscillations in these regions unless the mesh size is very small and thus the resulting algebraic system to be solved is considerably large. Part of this problem can be circumvented by modifying the variational formulation and adding stabilization terms but oscillations may remain or overshooting (or undershooting) phenomena may appear. In [9], stabilization methods are combined with graded meshes adapted to the solution to improve the numerical solution but using such hand-tailored meshes necessitates a precise knowledge of the solution. This is highly unrealistic for more general applications such as fluid flow problems in several space dimensions. In our numerical examples, we show that the automatic adaptation strategy presented here, which does not use an *a priori* knowledge of the solution (nor of the PDEs being solved) compares favorably with these hand-made meshes. Finally, we also test our strategy for approximating solutions presenting discontinuities, an even more challenging problem.

The major advantage of the one-dimensional case is that it can be pushed to the limit without unduly increasing the computational burden. As we shall see in the numerical results, we end up with elements of length as small as 10^{-11} in a unit domain. This would hardly be possible in 2D and even less in 3D.

This article is organized as follows. In Section 2 we introduce the one-dimensional convection-diffusion equation and its standard Galerkin and stabilized Petrov-Galerkin formulations. The error estimator and its interpretation as an interpolation error are presented in details in Section 3. In Section 4, each step and the whole methodology of the adaptation strategy is presented in the one-dimensional setting. In Section 5 we present our numerical tests, which compare our mesh adaptation strategy to uniform meshes or to hand-made graded meshes for convection-diffusion problems presenting boundary layers or even discontinuous solutions.

2. Position of the problem

As a test problem, we consider the classical convection-diffusion-reaction equation of the general form

$$-\frac{d}{dx}\left(d(x)\frac{du}{dx}(x)\right)+b(x)\frac{du}{dx}(x)+c(x)u(x)=f(x) \tag{1}$$

on the domain $\Omega=[\alpha,\beta]$ with Dirichlet boundary conditions $u(\alpha)=u_\alpha$ and $u(\beta)=u_\beta$. More general boundary conditions can also be considered. Convection-diffusion problems have been studied for a long time since it is well known that standard Galerkin finite element formulations (or centered finite differences) are unable to produce appropriate solutions when advection is dominant, unless very fine meshes are used. Some form of stabilization is needed, such as the Streamline-Upwind-Petrov-Galerkin (SUPG) formulation which goes back to the work of Hughes and Brooks [10, 11].

We will solve equation (1) by the finite element method. This requires a mesh $\mathcal{T}_h$ which is a partition of the interval $[\alpha,\beta]$ with N elements denoted $K=[x_i,x_{i+1}]$ with length $h_K=x_{i+1}-x_i$. Let V_h the space of Lagrange continuous functions