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Tailored Finite Point Method for Numerical Solutions of Singular Perturbed Eigenvalue Problems

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Abstract. We propose two variants of tailored finite point (TFP) methods for discretizing two dimensional singular perturbed eigenvalue (SPE) problems. A continuation method and an iterative method are exploited for solving discretized systems of equations to obtain the eigen-pairs of the SPE. We study the analytical solutions of two special cases of the SPE, and provide an asymptotic analysis for the solutions. The theoretical results are verified in the numerical experiments. The numerical results demonstrate that the proposed schemes effectively resolve the delta function like of the eigenfunctions on relatively coarse grid.

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1 Introduction

Consider the following eigenvalue problem

$$-\varepsilon^2\Delta\psi(\mathbf{x})+V(\mathbf{x})\psi(\mathbf{x})=\lambda\psi(\mathbf{x}),\quad \forall\mathbf{x}\in\mathbf{R}^n, \tag{1.1}$$

where $\psi(\mathbf{x})\to 0$ as $|\mathbf{x}|\to\infty$, $n=1,2,3$, $\varepsilon^2>0$ is a small diffusion coefficient or the half of the square of Planck constant $\hbar^2/2$, and $V(\mathbf{x})\geq 0$ is a given trapping potential function. This

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problem describes the wave function of one free particle under some nonnegative potential $V(\mathbf{x})$ (see [17, pp. 143]). The function ψ represents a quantity of the wave function of the quantum system, and $|\psi|$ is the probability amplitude with

$$\int_{\mathbf{R}^n} |\psi|^2 d\mathbf{x} = 1,$$

the mass conservation constraint for the wave function. Our concern in this paper is to study the analytic and numerical solutions of the eigenvalues and eigenfunctions of the SPE when ε^2 is small.

In [2], Ávila and Jeanjean studied a singular perturbed convection-reaction problem, $-\varepsilon^2\Delta u + V(\mathbf{x})u = f(u)$, and showed that the solution $u \in H^1(\mathbf{R}^n)$ concentrates at $\mathbf{x}_0$ where is the local minimum of the potential $V(\mathbf{x})$ when ε^2 approaches to zero. In this paper, we provide a mathematical analysis on the eigen-pairs of the SPE for two special cases: a constant potential and a harmonic potential, namely $V = |\mathbf{x}|^2/2$, and show that the square of eigenfunction of the SPE with a harmonic potential converges to a Dirac delta function weakly as ε^2 approaches to zero. In such case, traditional discretization methods such as central difference method and Galerkin finite element method yield inaccurate oscillatory solutions around the steep gradients.Thus, it is interesting to consider the design of robust and accurate scheme for solving the SPE numerically, whereas the solution contains steep gradients.

In [5, 6, 9], the tailored finite point (TFP) method was first proposed for the numerical solutions of singular perturbation problems with boundary layers. Later the TFP method has systematically been implemented for convection-dominated convection diffusion problems [7, 8, 10–14, 18, 19]. The TFP method gives an accurate computation of some features of the solution, particularly for small ε, without requiring a small meshsize. In this paper, we propose and implement two variants of TFP methods for solving the SPE. The numerical results are compared with those obtained from finite element method (FEM) to show the robust of our schemes.

This paper is organized as follows. In Section 2, we provide an asymptotical analysis for the eigen-pairs of the SPE in unbounded domain and bounded domain for some special cases. In Section 3, we derive two variants of TFP schemes for solving the SPE. In Section 4, we propose a continuation method and an iterative method for solving the discretization system of equations discretized by two variant TFP schemes, respectively. In Section 5, we examine the results of numerical experiments and demonstrate the robustness and accuracy of our proposed numerical methods.

2 The eigenvalue problem with singular perturbation

2.1 The singular perturbed eigenvalue problem on a unbounded domain