CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2022-0018

Vol. **5**, No. 2, pp. 320-349
June 2024

A C^1-Conforming Gauss Collocation Method for Elliptic Equations and Superconvergence Analysis Over Rectangular Meshes

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Received 14 May 2022; Accepted 23 January 2024

Abstract. This paper is concerned with a C^1-conforming Gauss collocation approximation to the solution of a model two-dimensional elliptic boundary problem. Superconvergence phenomena for the numerical solution at mesh nodes, at roots of a special Jacobi polynomial, and at the Lobatto and Gauss lines are identified with rigorous mathematical proof, when tensor products of C^1 piecewise polynomials of degree not more than $k, k \geq 3$ are used. This method is shown to be superconvergent with $(2k-2)$-th order accuracy in both the function value and its gradient at mesh nodes, $(k+2)$-th order accuracy at all interior roots of a special Jacobi polynomial, $(k+1)$-th order accuracy in the gradient along the Lobatto lines, and k-th order accuracy in the second-order derivative along the Gauss lines. Numerical experiments are presented to indicate that all the superconvergence rates are sharp.

AMS subject classifications: 65N12, 65N15, 65N30
Key words: Hermite interpolation, C^1-conforming, superconvergence, Gauss collocation methods, Jacobi polynomials.

1 Introduction

The C^1-conforming Gauss collocation method, also known as orthogonal spline collocation (OSC) method or spline collocation at Gauss points, was first proposed and studied by de Boor and Swartz [21] for solving two-point boundary value problems. Since then, considerable advances have been made in the formulation, analysis and application of

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this method, especially OSC for partial differential equations such as elliptic equations, initial-boundary value problems for parabolic, hyperbolic and Schrödinger-type systems (see, e.g. [22,25,28–30]) and so on. Comparing with the counterpart C^1-conforming finite element method, the most attractive feature of the C^1 collocation method is the simple and fast calculation of the coefficients of the mass and stiffness matrices since no integrals need to be evaluated or approximated, as well as its desired superconvergence phenomena not shared by the C^1 finite element method. Compared to the C^0 type such as finite volume methods (FVMs) and finite element methods (FEMs) or L^2 type such as the discontinuous Galerkin (DG) method, the advantage of the C^1-conforming method lies in the continuity of the first-order derivative approximation across the element interface and the higher order approximation in the second-order derivative approximation, with the same or less degrees of freedom.

There are some theoretical a priori results for the C^1-conforming Gauss collocation method in the literature, we refer to [4,6,26,27] for an incomplete list of reference. In [4,27] the authors analyzed the C^1-conforming Gauss collocation method for two dimensional elliptic equations on the rectangular mesh and established existence, uniqueness of the numerical solution, and derived optimal error estimates in the H^2, H^1 and L^2-norms. Meanwhile, superconvergence property of the method has also been investigated. It was proved in [5] that the solution of the C^1-conforming Gauss collocation method for the two-point boundary value problem is superconvergent at nodes with an order of $\mathcal{O}(h^{2k-2})$. As for two dimensional elliptic problems, it was observed numerically in [7,8] that the gradient value at the mesh nodes on rectangles has the same convergence rate $\mathcal{O}(h^{2k-2})$. However, a theoretical proof of this remarkable property remains open. Only for a very special case, i.e. $k=3$ on uniform rectangular meshes, the authors in [3,5] proved a fourth-order accuracy for the gradient approximation at mesh nodes. Comparing with other numerical methods such as FEMs (see, e.g. [2,9,23,24,31,33]), FVMs (see, e.g. [10,14,16,19,34]), DG methods (see, e.g. [1,13,15,18,35]), spectral Galerkin methods (see, e.g. [36,37]) in the literature, the superconvergence study for the C^1-conforming Gauss collocation methods is far from satisfied and developed.

The main purpose of our current work is to present a full picture for superconvergence properties of the C^1 collocation method for second-order elliptic problems in the two-dimensional setting. We prove that the method achieves convergence rate $2k-2$ for both solution and its gradient at mesh nodes under quasi-uniform rectangular meshes for piecewise bi-k polynomial space. In other words, we extend the superconvergence results in [3,5] from a special case (i.e. $k=3$ on uniform rectangular meshes) to a more general case (i.e. any polynomial $k \geq 3$ on non-uniform rectangular meshes). In addition, some new superconvergence points and lines are discovered, which are identified as Lobatto and Gauss lines and roots of a generalized Jacobi polynomial. To be more precise, we prove that the method is superconvergent with order $k+2$ at roots of a generalized Jacobi polynomial for the solution approximation; with order $k+1$ at interior Lobatto lines for the gradient approximation; with order k at Gauss lines for the second-order derivative approximation. As a byproduct, a supercloseness result of the numerical solution