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On the Well-Posedness of UPML Method for Wave Scattering in Layered Media

Wangtao Lu[1], Jun Lai[1,*] and Haijun Wu[2]

[1] *School of Mathematical Sciences, Zhejiang University, Zhejiang 310027, P.R. China.*
[2] *Department of Mathematics, Nanjing University, Jiangsu 210093, P.R. China.*

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Abstract. This paper proposes a novel method to establish the well-posedness of uniaxial perfectly matched layer (UPML) method for a two-dimensional acoustic scattering from a compactly supported source in a two-layered medium. We solve a long standing problem by showing that the truncated layered medium scattering problem is always resonance free regardless of the thickness and absorbing strength of UPML. The main idea is based on analyzing an auxiliary waveguide problem obtained by truncating the layered medium scattering problem through PML in the vertical direction only. The Green function for this waveguide problem can be constructed explicitly based on the separation of variables and Fourier transform. We prove that such a construction is always well-defined regardless of the absorbing strength. The well-posedness of the fully UPML truncated scattering problem follows by assembling the waveguide Green function through periodic extension.

AMS subject classifications: 35J05, 35J08, 74J20, 78A40

Key words: Helmholtz equation, perfectly matched layer, layered medium scattering, source scattering problem.

1 Introduction

Large amount of applications in optics (electromagnetics) and acoustics require the accurate analysis of wave scattering in layered media. Examples include optical waveguides, near field imaging, communication with submarine, detection of buried objects and so on. As a result, the analysis and numerical computation of layered medium scattering problems have been constantly attracting attentions from researchers both in engineering and mathematical communities [3,18,23,30].

*Corresponding author. *Email addresses:* `wangtaolu@zju.edu.cn` (W. Lu), `wangtaolu@zju.edu.cn` (J. Lai), `hjw@nju.edu.cn` (H. Wu)

In this paper, we are concerned with a two dimensional time harmonic acoustic scattering in a two-layered medium

$$\Delta u+k(\mathbf{x})^2u=f \quad \text{in } \mathbb{R}^2\backslash\Gamma, \tag{1.1}$$

where f is a source term with a compact support $D\in\mathbb{R}^2$, and u is the scattered field, as shown in Fig. 1(a). Denote by $\mathbf{x}=(x_1,x_2)$ the two dimensional coordinates. The interface Γ is simply assumed to be the axis $x_2=0$, by which the domain $\mathbb{R}^2$ is divided into the upper half space $\mathbb{R}^2_+$ and lower half $\mathbb{R}^2_-$, respectively. The wavenumber $k(\mathbf{x})$ takes the form

$$k(\mathbf{x})=\begin{cases} k_1, & \mathbf{x}\in\mathbb{R}^2_+, \\ k_2, & \mathbf{x}\in\mathbb{R}^2_-, \end{cases} \tag{1.2}$$

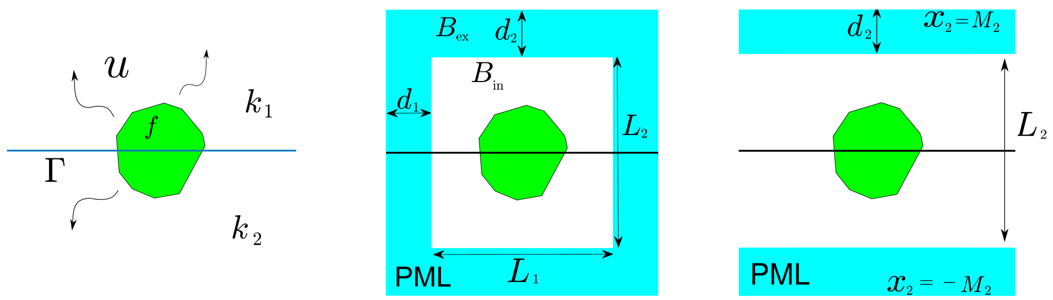
where k_1 and k_2 are two positive constants. We assume the field and flux are continuous across the interface Γ,

$$[u]_\Gamma=0, \quad [\partial_n u]_\Gamma=0, \tag{1.3}$$

where $[\cdot]$ denotes the jump on Γ. The scattered field u also satisfies the Sommerfeld radiation condition at infinity

$$\lim_{r\to\infty}\sqrt{r}\left(\frac{\partial u}{\partial r}-\mathbf{i}k(\mathbf{x})u\right)=0, \quad r=|\mathbf{x}|. \tag{1.4}$$

Due to important roles they play in applications, the layered medium scattering problems have been studied extensively in the literature. We refer readers to [2, 25] for the well-posedness of the acoustic scattering problems in a two-layered medium with locally perturbed interfaces and to [21] for the well-posedness of layered electromagnetic scattering problems. Discussions on the inverse scattering problems in a layered medium can be found in [3]. For numerical computations, given the infinite domain of Eq. (1.1), integral equation method is a natural candidate as they discretize the support D alone



(a) (b) (c)

Figure 1: The two-layered medium scattering problem with a compactly supported source f. (a) The original scattering problem. (b) The scattering problem with a full UPML truncation. (c) The waveguide problem with two infinitely long UPMLs on the top and bottom.