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On the Monotonicity of Q^3 Spectral Element Method for Laplacian

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Abstract. The monotonicity of discrete Laplacian, i.e., inverse positivity of stiffness matrix, implies discrete maximum principle, which is in general not true for high order accurate schemes on unstructured meshes. On the other hand, it is possible to construct high order accurate monotone schemes on structured meshes. All previously known high order accurate inverse positive schemes are or can be regarded as fourth order accurate finite difference schemes, which is either an M-matrix or a product of two M-matrices. For the Q^3 spectral element method for the two-dimensional Laplacian, we prove its stiffness matrix is a product of four M-matrices thus it is unconditionally monotone. Such a scheme can be regarded as a fifth order accurate finite difference scheme. Numerical tests suggest that the unconditional monotonicity of Q^k spectral element methods will be lost for $k \geq 9$ in two dimensions, and for $k \geq 4$ in three dimensions. In other words, for obtaining a high order monotone scheme, only Q^2 and Q^3 spectral element methods can be unconditionally monotone in three dimensions.

AMS subject classifications: 65N30, 65N06, 65N12

Key words: Inverse positivity, discrete maximum principle, high order accuracy, monotonicity, discrete Laplacian, spectral element method.

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1 Introduction

1.1 Monotone high order schemes

In many applications, monotone discrete Laplacian operators are desired and useful for ensuring stability such as discrete maximum principle [7] or positivity-preserving of physically positive quantities [15,24,28]. Let Δ_h denote the matrix representation of a discrete Laplacian operator, then it is called *monotone* if $(-\Delta_h)^{-1}\geq 0$, i.e., the matrix $(-\Delta_h)^{-1}$ has nonnegative entries. In this paper, all inequalities for matrices are entry-wise inequalities. It is well known that the simplest second order accurate centered finite difference scheme

$$u''(x_i)\approx\frac{u(x_{i-1})-2u(x_i)+u(x_{i+1})}{\Delta x^2}$$

is monotone because the corresponding matrix $-\Delta_h$ is an M-matrix thus inverse positive. The most general extension of this result is to state that a linear finite element method with special implementation under a mild mesh constraint forms an M-matrix thus monotone on unstructured triangular meshes [32]. Other than discrete maximum principle, monotonicity often implies more properties, e.g., energy dissipation can be proven via Jensen's inequality for a monotone scheme solving a Keller-Segel equation [15].

The discrete maximum principle is not true for high order finite element methods on unstructured meshes [14]. On structured meshes, there exist a few high order accurate inverse positive finite difference schemes. To the best of our knowledge, the following schemes for solving a two-dimensional Poisson equation are the only ones proven to be monotone beyond the second order accuracy, and all of them can be regarded as finite difference schemes with four order accuracy for function values for solving elliptic and parabolic equations:

1. The classical 9-point scheme [4,10,17] are monotone because the stiffness matrix is an M-matrix.

2. In [3,5], a fourth order accurate finite difference scheme was constructed. The stiffness matrix is a product of two M-matrices thus monotone.

3. The Lagrangian P^2 finite element method on a regular triangular mesh [31] has a monotone stiffness matrix [25]. On an equilateral triangular mesh, the discrete maximum principle can also be proven [14]. It can be regarded as a finite difference scheme at vertices and edge centers, on which superconvergence of fourth order accuracy holds.