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Generalized Isospectral-Nonisospectral Modified Korteweg-de Vries Integrable Hierarchies and Related Properties

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Abstract. In this article, a new technique for deriving integrable hierarchy is discussed, i.e., such that are derived by combining the Tu scheme with the vector product. Several classes of spectral problems are introduced by three-dimensional loop algebra and six-dimensional loop algebra whose commutators are vector product, and the six-dimensional loop algebra is derived from the enlargement of the three-dimensional loop algebra. It is important that we make use of the variational method to create a new vector-product trace identity for which the Hamiltonian structure of the isospectral integrable hierarchy is worked out. The derived integrable hierarchies are reduced to the modified Korteweg-de Vries (mKdV) equation, generalized coupled mKdV integrable system and non-isospectral mKdV equation under specific parameter selection. Starting from a 3×3 matrix spectral problem, we subsequently construct an explicit N-fold Darboux transformation for integrable system (2.8) with the help of a gauge transformation of the corresponding spectral problem. At the same time, the determining equations of nonclassical symmetries associated with mKdV equation are presented in this paper. It follows that we investigate the coverings and the nonlocal symmetries of the nonisospectral mKdV equation by applying the classical Frobenius theorem and the coordinates of a infinitely-dimensional manifold in the form of Cartesian product.

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1 Introduction

It has been a hot topic in searching for new integrable systems. Since Magri [1] proposed the Lax pair method for generating integrable equations, a lot of experts and scholars have devoted them to this research field so that a variety of modified methods have been developed in the past few years [2–8]. One of the most classical method [6] called Tu scheme by Ma [7] applies Lax pairs adjoint with finite-dimensional Lie algebras to investigate integrable equations. Before we present the scheme, a few of basic notations are first introduced.

Let G be a matrix Lie algebra over the complex field C and $\tilde{G}=G\otimes C(\lambda,\lambda^{-1})$ be its loop algebra, where $C(\lambda,\lambda^{-1})$ is the set of Laurent polynomials in λ. The gradation of $\tilde{G}$ is taken by $deg(x\otimes\lambda^n)=n, x\in G$. Let $g\in\tilde{G}$ and $g=\sum_n g_n$, $degg_n=n$, be its gradation decomposition. Set $g_+=\sum_{n\geq 0}g_n$, we consider the isospectral problem

$$\varphi_x=U(u,\lambda)\varphi,$$

with $U=U(u,\lambda)=e_0(\lambda)+u_1e_1(\lambda)+\cdots+u_pe_p(\lambda)$, where $u=(u_1,\cdots,u_p)$ is a potential function, $e_0(\lambda),e_1(\lambda),\cdots,e_p(\lambda)\in\tilde{G}$. Suppose $e_0,e_1,\cdots,e_p$ are linearly independent and $\varepsilon_0>0$, $\varepsilon_0>\varepsilon_i$, $i=1,\cdots,p$; here $\varepsilon_i=dege_i$. The explicit steps of Tu scheme for generating Lax integrable systems are as follows:

First, we take a solution $V=V(\lambda)$ of the equation

$$V_x(\lambda)=[U(\lambda),V(\lambda)].$$

Second, we search for $\Delta_n\in\tilde{G}$ so that, for $V^{(n)}=(\lambda^nV)_++\Delta_n$, it holds that

$$V_x^{(n)}-[U,V^{(n)}]=Ce_1+\cdots+Ce_p.$$

This requirement yields a hierarchy of evolution equations

$$U_{t_n}=V_x^{(n)}-[U,V^{(n)}]. \tag{1.1}$$

Finally, using the trace identity

$$\frac{\delta}{\delta u_i}\left\langle V,\frac{\partial U}{\partial\lambda}\right\rangle=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left\langle V,\frac{\partial U}{\partial u_i}\right\rangle$$

deduces the generalized Hamiltonian structure of (1.1), where $\langle x,y\rangle=tr(xy)$, $x,y\in\tilde{G}$.