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Geometric Decomposition and Efficient Implementation of High Order Face and Edge Elements

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Abstract. This study investigates high-order face and edge elements in finite element methods, with a focus on their geometric attributes, indexing management, and practical application. The exposition begins by a geometric decomposition of Lagrange finite elements, setting the foundation for further analysis. The discussion then extends to $H(\mathrm{div})$-conforming and $H(\mathrm{curl})$-conforming finite element spaces, adopting variable frames across differing sub-simplices. The imposition of tangential or normal continuity is achieved through the strategic selection of corresponding bases. The paper concludes with a focus on efficient indexing management strategies for degrees of freedom, offering practical guidance to researchers and engineers. It serves as a comprehensive resource that bridges the gap between theory and practice.

AMS subject classifications: 65N30, 35Q60

Key words: Implementation of finite elements, nodal finite elements, $H(\mathrm{curl})$-conforming, $H(\mathrm{div})$-conforming.

1 Introduction

This paper introduces node-based basis functions for high-order finite elements, specifically focusing on Lagrange, BDM (Brezzi-Douglas-Marini) [8, 9, 18], and second-kind Nédélec elements [6, 18]. These elements are subsets of the spaces H^1, $H(\mathrm{div})$, and

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$H(\mathrm{curl})$, with their shape functions being the full polynomial space $\mathbb{P}_k^n$, where k represents the polynomial degree, and n the geometric dimension. Notably, varying continuity across these elements gives rise to distinct characteristics. When $n=3$, for the lowest degree $k=1$, degrees of freedom (DoFs) of the $H(\mathrm{div})$-conforming finite element are posed on faces and DoFs of the $H(\mathrm{curl})$-conforming finite element are posed on edges. Therefore, conventionally an $H(\mathrm{div})$-conforming finite element is referred to as a face element, and an $H(\mathrm{curl})$-conforming element is an edge element. They are also known as the second family of face and edge element as the shape function space is the full polynomial space while the first family consists of incomplete polynomial spaces [6].

In the realm of Lagrange finite elements, nodal basis functions stand out for their simplicity and ease of computation. In contrary, constructing basis functions for face and edge elements is more intricate. Traditional approaches involve the Piola transformations, where basis functions are first devised on a reference element and subsequently mapped to the actual element using either covariant (to preserve tangential continuity, in the case of edge elements) or contravariant (to maintain normal continuity, for face elements) Piola transformations. Detailed explanations of this approach can be found in [17, 20], and implementation is in open-source software such as MFEM [5] and Fenics [4].

Arnold, Falk and Winther, in [7], introduced a geometric decomposition of polynomial differential forms. Basis functions based on Bernstein polynomials were proposed, paving the way for subsequent advancements. In [1,3], basis functions founded on Bernstein polynomials were explored, accompanied by fast algorithms for the matrix assembly. Additionally, hierarchical basis functions for $H(\mathrm{curl})$-conforming finite elements were introduced in [2, 22–24].

While these methods offer valuable insights, they can be quite complex. Researchers have thus ventured into simpler approaches. In [14], a method multiplying scalar nodal finite element methods by vectors was introduced, resulting in $H(\mathrm{div})$ and $H(\mathrm{curl})$ conforming finite elements that exhibit continuity on both vertices and edges.

We propose a straightforward method to construct nodal bases for the second family face and edge elements. Initially, we clarify the basis of Lagrange elements by considering them dual to the degrees of freedom, which are determined by values at interpolation points. We then extend this principle to vector polynomial spaces, wherein each interpolation point establishes a frame that includes both tangential and normal (t-n) directions. We impose the continuity of either tangential or normal components by appropriate choice of the t-n decomposition. We explicitly derive the dual basis functions for these elements from the basis functions of Lagrange elements.

This idea has been previously explored in [10, 15, 16] for constructing a hierarchical basis of $H(\mathrm{div})$ elements in two and three dimensions. Through a rotation process, it was also adapted for $H(\mathrm{curl})$ elements in two dimensions, as detailed in [15, 16]. However, extending their methodology to higher dimensions presented significant challenges. Our work advances this field by developing a geometric decomposition of the second family face and edge elements in arbitrary dimensions and orders. Additionally, we introduce