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MODIFIED NEWTON-NDSS METHOD FOR SOLVING NONLINEAR SYSTEM WITH COMPLEX SYMMETRIC JACOBIAN MATRICES

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Abstract. An efficient iteration method is provided in this paper for solving a class of nonlinear systems whose Jacobian matrices are complex and symmetric. The modified Newton-NDSS method is developed and applied to the class of nonlinear systems by adopting the modified Newton method as the outer solver and a new double-step splitting (NDSS) iteration scheme as the inner solver. Additionally, we theoretically analyze the local convergent properties of the new method under the weaker Hölder conditions. Lastly, the new method is compared numerically with some existing ones and the numerical experiments solving the nonlinear equations demonstrate the superiority of the Newton-NDSS method.

Key words. Modified Newton-NDSS method, complex nonlinear systems, Hölder continuous condition, symmetric Jacobian matrix, convergence analysis.

1. Introduction

Consider the complex nonlinear systems with the following form

$$F(x) = 0, \tag{1}$$

with $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ representing a nonlinear function. Further, the function F is defined on an open convex subset $\mathbb{D}$ of the n- dimensional complex linear space $\mathbb{C}^n$ and continuously differentiable. For the sake of solving the nonlinear systems with effectiveness, we first review the study of the solution technique to linear systems with complex matrices

$$Az = b, \quad A = W + iT \in \mathbb{C}^{n\times n}, \quad z, b \in \mathbb{C}^n. \tag{2}$$

Here the matrices $W, T \in \mathbb{R}^{n\times n}$ are real symmetric and positive semidefinite with at least one of them being positive definite. Here the matrices $W, T \in \mathbb{R}^{n\times n}$ are real symmetric and positive semidefinite with at least one being positive definite. Throughout this paper, $i = \sqrt{-1}$ defines the imaginary unit. Systems (2) appear in a variety of engineering applications and scientific computing, such as diffuse optical tomography, structural dynamics, and quantum mechanics. Readers can refer to the literature [1-3]. Up to now, researchers have made great efforts to seek rapid solution techniques for the above complex linear systems (2). From the very beginning, Bai et al. originated the classical Hermitian and skew-Hermitian splitting (HSS) [4] iteration method and its preconditioned form PHSS [5] method for semidefinite linear or positive definite systems. Afterward, the modified HSS (MHSS) method [6] iteration scheme were constructed by Bai et al. which greatly enhanced the computational efficiency of HSS iteration scheme. Whereafter, variations and improvements of these methods proliferated [7-11]. Especially, various numerical methods have been produced through internal and external iterative techniques. The solution of the large sparse positive definite system of nonlinear equations has

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been studied by Bai et al. [12], who developed the Newton-HSS methods. For the large sparse systems with complex symmetric Jacobian matrices, Yang and Wu [13] applied the inexact Newton-MHSS method to them. Therewith, the research on large sparse nonlinear systems attracted substantial attention [14-18]. Easily stated, the following block system is equivalent to (2) and can avoid the operations on complex matrices, so that attracted a vast scale of interest.

$$\mathcal{A}x = \begin{bmatrix} W & -T \\ T & W \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{3}$$

where $z = u + iv$ and $b = p + iq$. Bai et al. [19] introduced a block preconditioned MHSS (PMHSS) iteration method [19] and its alternating-directional versions [20] for the above linear system (3). Taking into account the excellent properties and efficient performance of SOR-like methods, over the recent years, the generalized SOR (GSOR) method [21], accelerated GSOR (AGSOR) method [22], and preconditioned GSOR (PGSOR) method [23] were applied to the two-by-two block linear system (3). A series of iteration schemes based on GSOR-like methods which can converge to the exact solution to complex nonlinear equations rapidly were proposed named modified Newton-GSOR method [24], modified Newton-AGSOR method [25], and modified Newton-PGSOR method [26]. A while back, a fixed-point iteration adding the asymptotical error (FPAE) scheme and its parameterized variant were structured by Xiao and Wang [27]. Then, a class of complex nonlinear systems has been solved by Zhang and Wu [28] using the Newton-FPAE method and the modified Newton-FPAE method. Recently, Huang [29] developed a new double-step splitting (NDSS) iteration method by taking advantage of two-step, parameter accelerating and preconditioning techniques. It has been proved that the NDSS iteration method demands mild convergence conditions and owns a faster convergence speed compared to some known iteration methods. These indicated the effectiveness and practicability of the NDSS method.

The aim of the present work is to formulate a fast and effective iterative method for solving complex nonlinear systems. Inspired by the excellent computing ability of the NDSS method compared with other algorithms for complex linear systems, we elaborate the modified Newton-NDSS (MN-NDSS) iteration method for the complex nonlinear systems by applying the modified Newton method as the outer solver and the NDSS method as the inner solver.

This paper is organized as follows. In the next section, we outline the modified Newton-NDSS iteration scheme for solving the complex nonlinear systems, including its algorithm and iterative formula. Section 3 is devoted to analyse the local convergent properties under the Hölder hypothesis for the new iteration methods. The results of numerical experiments presented in Section 4 support the theoretical findings and explain the superiority of the modified Newton-NDSS method. Finally, some conclusions are given in Section 5.

2. The modified Newton-NDSS method

First, throughout the paper, denote

$$\mathcal{A} = \begin{bmatrix} W & -T \\ T & W \end{bmatrix}, \quad x = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{and} \quad \tilde{b} = \begin{bmatrix} p \\ q \end{bmatrix},$$

with $W, T \in \mathbb{R}^{n\times n}$ being symmetric positive semidefinite and at least one of them being positive definite. Obviously, system (3) can be reformulated as

$$\tilde{\mathcal{A}}x = \begin{bmatrix} T & W \\ -W & T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} q \\ -p \end{bmatrix}. \tag{4}$$