INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 21, Number 2, Pages 268–294

doi: 10.4208/ijnam2024-1011

RICHARDSON EXTRAPOLATION OF THE CRANK-NICOLSON SCHEME FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this work, we consider Richardson extrapolation of the Crank-Nicolson (CN) scheme for backward stochastic differential equations (BSDEs). First, applying the Adomian decomposition to the nonlinear generator of BSDEs, we introduce a new system of BSDEs. Then we theoretically prove that the solution of the CN scheme for BSDEs admits an asymptotic expansion with its coefficients the solutions of the new system of BSDEs. Based on the expansion, we propose Richardson extrapolation algorithms for solving BSDEs. Finally, some numerical tests are carried out to verify our theoretical conclusions and to show the stability, efficiency and high accuracy of the algorithms.

Key words. Backward stochastic differential equations, Crank-Nicolson scheme, Adomian decomposition, Richardson extrapolation, asymptotic error expansion.

1. Introduction

This paper is concerned with the numerical solution of the following BSDE defined on a filtered complete probability space $(\Omega,\ \mathcal{F},\ \mathbb{F},\ \mathbb{P})$ with the natural filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ generated by a standard d_1-dimensional Brownian motion $W_t = (W_t^1, W_t^2, \cdots, W_t^{d_1})^\top$, $0 \le t \le T$.

$$Y_t = \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \tag{1}$$

where T is a deterministic terminal time instant; $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^q$ and $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R}^{q\times d_1} \longrightarrow \mathbb{R}^q$ are the terminal condition and the generator of BSDE (1), respectively. Note that the stochastic integral with respect to W_t is of Itô's type, and X_t is a diffusion process. In this paper, we only consider the case where

$$X_t = X_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s,\ 0 \le t \le T, \tag{2}$$

where the functions $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d_1}$ are called the drift and the diffusion coefficients of the SDE (2). A pair of processes $(Y_t,\ Z_t)$ is called an L^2-adapted solution of (1) if it is $\mathcal{F}_t$-adapted, square integrable, and satisfies BSDE (1).

In 1990, the existence and uniqueness of the solution of BSDEs were proved by Pardoux and Peng [28]. Since then, lots of efforts have been devoted to the study of BSDEs due to their applications in various important fields such as mathematical finance, stochastic optimal control, risk measure, game theory, and so on (see, e.g., [12, 32, 26, 29] and references therein).

As BSDEs seldom admit explicitly closed-form solutions, numerical methods have played an important role in applications. In recent years, great efforts have been made for designing efficient numerical schemes for BSDEs and forward

Received by the editors on November 30, 2023 and, accepted on January 22, 2024.
2000 *Mathematics Subject Classification.* 65C30, 60H10, 60H35.
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backward stochastic differential equations (FBSDEs). There are two main types of numerical schemes: the first one is based on numerical solution of a parabolic PDE which is related to a FBSDE [11, 25], while the second type of schemes focus on discretizing FBSDEs directly [3, 5, 10, 18, 24, 33, 40]. From the temporal discretization point of view, popular strategies include Euler-type methods [15, 16, 38], θ-schemes [34, 43], Runge-Kutta schemes [8], multistep schemes [7, 14, 41, 44, 45], and strong stability preserving multistep (SSPM) schemes [13], to name a few. For fully coupled FBSDEs, there exist only few numerical studies and satisfactory results [27, 41]. We mention the work in [41], where a class of multistep type schemes are proposed, which turns out to be effective in obtaining highly accurate solutions of FBSDEs, and the work in [35], where the classical deferred correction (DC) method is adopted to design highly accurate numerical methods for fully coupled FBSDEs.

In this paper, we will approximate the solution of BSDE (1) based on the Richardson extrapolation (RiE) method. It is well known that Richardson extrapolation method, which was established by Richardson [31], is an efficient procedure for increasing the accuracy of approximations of many problems in numerical analysis. For example, the applications of the RiE to ordinary differential equations (ODEs) based on one-step schemes, e.g., Runge-Kutta methods are described in [6, 17]. In addition, this method has been well demonstrated in its applications to finite element and mixed finite element methods for elliptic partial differential equations [4], Sobolev- and viscoelasticity-type equations [22], partial integro-differential equations [23], Fredholm and Volterra integral equations of the second kind [20], Volterra integro-differential equations [39], and to collocation methods in [21], etc. As for the applications of the RiE to BSDEs, we mention the work in [9], where an explicit error expansion for the solution of BSDEs is obtained by using the cubature on Wiener spaces method.

In this work, we will design highly accurate Richardson extrapolation algorithms with the solutions of the Crank-Nicolson scheme for BSDE (1). To this end, we first introduce a new system of BSDEs by applying the Adomian decomposition to the nonlinear generator of BSDEs. Then we theoretically prove that the solution of the Crank-Nicolson scheme for BSDEs admits an asymptotic expansion with its coefficients being the solutions of the new system of BSDEs. Finally, based on the expansion, we propose the Richardson extrapolation algorithms of the Crank-Nicolson scheme (RiE-CN, for short) for solving BSDEs. The RiE-CN algorithms are very easy in use. We can obtain accurate solutions with high order rate of convergence only by combining linearly the numerical solutions of the CN scheme with different time step sizes. Moreover, our numerical tests verify our theoretical conclusions, and show that the RiE-CN algorithms are stable, very efficient and high accurate.

The rest of the paper is organized as follows. In Section 2, we recall the nonlinear Feynman-Kac formula, the generator of a diffusion process, the Adomian decomposition and the Richardson extrapolation method in brief. We present the asymptotic error expansion of the solution of the Crank-Nicolson scheme for BSDEs in Section 3. The construction of the RiE-CN algorithms for BSDEs is presented in Section 4. And in Section 5, numerical tests are carried out to support the theoretical results. Finally, some concluding remarks are given in Section 6.