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A HYBRID STRESS FINITE ELEMENT METHOD FOR INTEGRO-DIFFERENTIAL EQUATIONS MODELLING DYNAMIC FRACTIONAL ORDER VISCOELASTICITY

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Abstract. We consider a semi-discrete finite element method for a dynamic model for linear viscoelastic materials based on the constitutive law of fractional order. The corresponding integro-differential equation is of a Mittag-Leffler type convolution kernel. A 4-node hybrid stress quadrilateral finite element is used for the spatial discretization. We show the existence and uniqueness of the semi-discrete solution, then derive some error estimates. Finally, we provide several numerical examples to verify the theoretical results.

Key words. Integro-differential equation, fractional order viscoelasticity, hybrid stress finite element, error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with boundary Γ, and let T be a positive constant. We consider a hyperbolic type integro-differential system arising in the theory of linear and fractional-order viscoelasticity:

$$(1)\qquad \begin{cases} \rho \mathbf{u}_{tt} - \mathrm{div}\boldsymbol{\sigma} = \mathbf{f}(\mathbf{x},t), & (\mathbf{x},t) \in \Omega \times (0,T], \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}_0 - \int_0^t K(t-s)\boldsymbol{\sigma}_0(\cdot,s)\mathrm{d}s, & (\mathbf{x},t) \in \Omega \times (0,T], \\ \mathbf{u} = 0, & (\mathbf{x},t) \in \Gamma \times (0,T], \\ \mathbf{u}(\mathbf{x},0) = \varphi_0(\mathbf{x}), \mathbf{u}_t(\mathbf{x},0) = \varphi_1(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases}$$

Here $\rho > 0$ is the (constant) mass density, $\mathbf{u}(\mathbf{x},t) = (u_1,u_2)^T$ the displacement field, $\boldsymbol{\sigma}(\mathbf{x},t) = (\sigma_{ij})_{2\times 2}$ the symmetric stress tensor, $\mathbf{f}(\mathbf{x},t)$ the body force, and $\varphi_0(\mathbf{x}), \varphi_1(\mathbf{x})$ the initial data. $\boldsymbol{\sigma}_0(\mathbf{x},t)$ denotes the elastic stress tensor,

$$\boldsymbol{\sigma}_0 := 2\mu\epsilon(\mathbf{u}) + \lambda \mathrm{tr}(\epsilon(\mathbf{u}))\mathbf{I},$$

with $\lambda, \mu > 0$ being the Lamé constants, $\epsilon(\mathbf{u}) = \dfrac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ the strain tensor, $\mathrm{tr} \doteq \mathrm{tr}(\cdot)$ the trace of a matrix, and $\mathbf{I}$ the 2×2 identity. For $0<\nu<1, 0<\alpha<1$, the convolution kernel

$$(2)\qquad K(t) := -\nu \frac{d}{dt} E_{\alpha,1}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right) = \nu \frac{t^{\alpha-1}}{\tau^{\alpha}} E_{\alpha,\alpha}\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right),$$

where $\tau > 0$ is the relaxation time, and

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

denotes the two-parameter Mittag-Leffler function.

Fractional order viscoelastic models are capable of accurately describing memory and non-locality properties of viscoelastic materials [3, 4, 5, 6, 10, 12, 13, 15, 16, 21, 34, 39]. In fact, the second equation of (1), which involves a convolution integral

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and is an explicit expression for the stress tensor in terms of the strain tensor, originates from the fractional-order viscoelastic constitutive law

$$\sigma + \tau^\alpha D_t^\alpha \sigma = (1-\nu)\boldsymbol{\sigma}_0 + \tau^\alpha D_t^\alpha \boldsymbol{\sigma}_0, \tag{3}$$

where

$$D_t^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) \mathrm{d}s.$$

denotes the left Riemann-Liouville operator of fractional differentiation of order α. The explicit expression is obtained by using Laplace transform techniques on (3), and the use of the convolution integral formulation avoids the difficulties concerning the physical interpretation, justification and verification of fractional order initial conditions; see [1, 11, 13, 14].

There are many works on the numerical analysis of related displacement models of (1) where the stress tensor $\boldsymbol{\sigma}$ does not appear as an independent variable; see, e.g. [2, 22, 23, 35, 37, 38]. Adolfsson et al. [2] and Saedpanah [35, 38] studied spatial semi-discrete continuous Galerkin finite element methods and gave optimal a priori error estimates. Larsson et al. [22] analyzed a temporal semi-discrete discontinuous Galerkin method based on piecewise constant polynomials. In [23, 37] Larsson and Saedpanah used the continuous space-time linear finite element method to formulate the full discretizations, and derived optimal error estimates. We also refer to [18, 19, 32, 33] for some literature on numerical treatment of linear viscoelasticity problems with exponential kernels in the constitute equation.

In the numerical analysis of elasticity, the hybrid stress finite element method, pioneered by Pian [28], is known to be an efficient approach to improve the performance of the standard 4-node compatible displacement quadrilateral (bilinear) element (cf. [28, 29, 30, 41, 42, 43, 45, 49, 50]). This method is based on the domain-decomposed Hellinger-Reissner variational principle, which includes the displacement and stress variables. Since the stress parameters can be eliminated at the element level, only the unknowns of the displacements will remain in the resulting final discrete system. In [30] Pian and Sumihara proposed a robust 4-node hybrid stress quadrilateral element by using a rational choice of the 5-parameter stress mode. In [43, 42, 47, 49] optimal stress modes were studied for two- and three-dimensional hybrid stress elements. We refer to [24, 45, 50] for the stability and convergence analysis of 4-node hybrid stress quadrilateral elements. In [44, 46] and [40] semi-discrete and fully discrete hybrid stress methods were proposed and analyzed for linear elastodynamic problems and Maxwell viscoelastic problems, respectively.

In this paper, we apply the hybrid stress finite element method to discretize the viscoelastic model (1) to obtain a spatial semi-discrete scheme. The standard isoparametric bilinear interpolation is used for the displacement approximation, and the Pian-Sumihara's 5-parameters stress mode is used for the stress approximation. We prove the existence and uniqueness of the semi-discrete solution, and derive optimal error estimates.

The rest of the paper is organized as follows. Section 2 introduces notations and weak formulations. Section 3 gives the semi-discrete hybrid stress scheme and carries out the error analysis. Finally, Section 4 provides some numerical results.

2. Notations and weak formulations

Throughout this paper,we use $H^r(\Omega)$ to denote the standard Sobolev spaces with norm $\|\cdot\|_r$ and semi-norm $|\cdot|_r$. And $H^0(\Omega) = L^2(\Omega)$ is the space of square integral